THE SYMMETRIC GROUP AND THE UNITARY GROUP : An Application of Group-Subgroup Transformation Theory.

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PREFACE

This thesis is the result of research into group theoretical techniques, in particular, the study of symmetry properties of 6j and 3jm symbols of the unitary group schemes. From earlier work, the numerical tables of 6j symbols for E_7, SU_6 and SU_3 and 3jm symbols of $E_7 \supset SU_6 \times SU_3$, $SU_6 \supset SU_2 \times SU_3$ and $SU_3 \supset U_1 \times SU_2$ (Butler, Haase and Wybourne 1978, 1979; Bickerstaff, Butler, Butts, Haase and Reid 1982) provided illustration of the successfulness of the Butler method in calculation of 6j and 3jm symbols and also rank independent properties of these symbols. Two distinct research topics arose from this work. From one, I have demonstrated that algebraic formulae for SU_n 6j symbols can be obtained using the Butler method. Several symmetries for the 6j symbols, in particular the transpose conjugate symmetry are apparent from the table. The results of this work have been presented for publication (Haase and Butler 1983a).

The second topic, which has been called the Schur-Weyl duality, uses the close relationship between the unitary groups and the symmetric groups. Many powerful relations are obtained and these extend the results given by Kramer (1967, 1968), Kramer and Seligman (1969a), Sullivan (1973, 1975) and Chen (1981) (see also Haase 1979). I have formulated the duality symmetry in three lemmas, each defining a duality factor. These factors have been omitted or assumed unity by previous authors. I have given the symmetries and phase freedom of the duality factors and have shown that in conjunction with the $U_n \simeq U_1 \times SU_n$ isomorphism, there is insufficient freedom to choose all duality factors, although many can be chosen the unit matrix. This does not however exclude the possibility that all duality factors are, in fact, unity. Much of this work has been accepted for publication (Haase and Butler 1983c) and a further paper on phase freedom and matrix choices of the duality factors is in preparation.

In establishing the many duality results, I have needed to derive new results in the theory of transformation coefficients for arbitrary groups. I have given a new perspective to the Racah-Wigner coupling algebra showing that its theorems are special cases of those in the group-subgroup transformation theory. Coupling and recoupling coefficients have been generalized. In the area of induced representations I have defined new transformation factors, the induction and reinduction factors. A paper presenting this work has been accepted for publication (Haase and Butler 1983b). In addition I have described, in analogy to the coupling factor and recoupling coefficient of the Racah-Wigner algebra, the symmetries and a calculational method of other transformation factors including the induction and reinduction factors. A paper on these aspects is in preparation.

To the following people who have helped me during my postgraduate years, I express my sincere gratitude. I owe much to my supervisor, Dr. P.H. Butler, for his continued guidance, support, and patience. I have received much help and encouragment from Professor B.G. Wybourne especially as my supervisor during Dr. Butler's term of leave. I have enjoyed fruitful discussions with Dr. G.R.E. Black and Dr. M.F. Reid. For his critical reading of this thesis and some additional commas I owe Dr. S.H. Payne many thanks. To my fellow research students, I thank them for their companionship especially Graeme and David for without whom the 5th floor would not have been the same. Mrs. Mary Boswell typed the manuscript with great ability and patience. Mrs. Janet Warburton has given her typing expertise to several papers. My warmest thanks go to them both.

Ric Haase,

February, 1983.

ABSTRACT

The general mathematical aspects of transformation theory of an arbitrary group-subgroup scheme are developed. The theorems of the Racah-Wigner coupling algebra are shown to be a special case of this general theory. Several new transformation factors are defined, and symmetries and a calculational method based on the Butler method are presented.

The Butler method is used to obtain n dependent algebraic formulae for some 6j symbols of the special unitary group SU_n. Combined with the composite labelling for SU_n irreps, several symmetries of these 6j formulae are apparent, in particular the transpose conjugate symmetry of the symmetric group.

The close relationship between the unitary group and the symmetric group are reviewed and many relations developed. In particular duality factors are defined and their symmetry and phase freedom properties are discussed. It is found that in connection with the $U_n \simeq U_1 \times SU_n$ isomorphism, insufficient phase freedom exists to choose all duality and isomorphism factors.

LIST OF TABLES

CHAPTER I

INTRODUCTION

Group theory is a widely used tool in physics. It is used in a variety of research fields from the well established fields of nuclear, solid state and molecular physics to the currently topical fields of elementary particles, lattice gauge theory and grand unified field theory. The part of group theory used in this thesis is called the Racah-Wigner algebra and it originates from a generalization of angular momentum theory (see Biedenharn and van Dam 1965). In the application of the various theorems of group theory, the Wigner-Eckart theorem is foremost. This theorem gives the matrix elements of a tensor operator in terms of a coupling coefficient. The basis dependent coupling coefficient and the basis independent recoupling coefficient can be replaced by the more symmetric 3jm and 6j symbols which were introduced by Wigner (1940). The Racah-Wigner algebra focuses on the symmetries and numerical evaluation of both 6j and 3jm symbols for various groups (Derome 1966, Derome and Sharp 1965, Butler 1975, 1981). A major concern of this thesis is the study of these symbols for the unitary groups. To facilitate our discussion we need to generalize the Racah-Wigner algebra. We call this generalization the group-subgroup (GH) transformation theory. Thus this thesis is divided into two parts:

(1) the general mathematical aspects of the GH transformation theory (Chapters II and III) and

(2) the application of the GH transformation theory to the unitary groups. Two separate aspects are considered:

(i) the use of the Butler method to obtain algebraic formulae for SU_n 6j symbols (Chapter IV), and

(ii) the symmetry of U_n transformation factors arising from the correspondence between the unitary group and the symmetric groups (Chapters V and VI). We call this relationship the Schur-Weyl duality.

In our development of the GH transformation theory several concepts are brought together. We start from the concept of the action of the group on a vector space. The action is known as the representation of the group and the vector space is known as a representation space (of the group). In general a representation space is reducible into minimal invariant subspaces or irreducible representation spaces (irrep spaces). This irreducibility concept implies that there exists no subspace of the irrep space in which the action of the group is invariant. In terms of the basis for the representation space, the action of the group is described by a set of matrices - a matrix representation. The basis of a fully reduced vector space is called an irrep basis. If any two irrep spaces are the same, we may require the irrep basis to give identical matrix representations for both irrep spaces. We call such an irrep basis a G basis for the set of all such irrep spaces. The G basis concept plays an important role in the GH transformation theory.

We are concerned with transformations between bases. Specifically, the transformation between (1) the basis obtained as the product basis of a tensor

product of two irrep spaces (of not necessarily the same group) and the irrep basis defines a coupling coefficient,
(2) a basis obtained by restriction to a subgroup and its irrep basis defines a subduction coefficient, and
(3) a basis obtained by induction and its irrep basis defines an induction coefficient.

By using particular G bases, Schur's lemmas and an (irrep) matrix invariance requirement, we define in Chapter II special types of transformation coefficients which we call transformation factors. These factors are independent of a subgroup basis labels in the same manner as the recoupling coefficient. Indeed, the well-known coupling factor and recoupling coefficient and Kramer's less well-known 6f symbols are all special cases of transformation factors. We introduce several generalized and some new transformation factors which we have called coupling and recoupling factors, subduction and resubduction factors, and induction and reinduction factors. In Chapter III, we give a systematic discussion of the complex conjugation, transposition and associativity symmetries of the various transformation factors. A method of calculation is proposed for these factors based on the recursive building-up method used by Butler to evaluate 6j and 3jm symbols of an arbitrary group.

As a specific example of the building-up method we have evaluated in Chapter IV 6j symbols of the special unitary group SU_n. The unitary groups were chosen for the fact that a variety of such groups have been used in many fields in physics. For this reason they will always be important in physics. These uses include SU_2 for

for angular momentum and isospin, SU_3 for the harmonic oscillator problem and colour charge in QCD, SU_4 for the Wigner supermultiplets in nuclear physics, SU_5 in grand unified field theories, SU_{10} and SU_{14} for strong and weak crystal field calculations and many more.

The building-up method, which is familiar to nuclear physicists, was shown by Butler and Wybourne (1976) to have the advantage of requiring only a knowledge of character theory (the 'mathematician's group theory'). This knowledge involves mainly product and branching rules. The method is particularly useful for groups with irreps of large dimension such as E_7 (Butler, Haase and Wybourne 1978, 1979) where other methods are impractical. It has been successfully applied to obtain algebraic formulae for SO_3 6j symbols and $SO_3 \supset SO_2$ 3jm symbols (Butler 1976), and numerical tables for the point group (Butler 1981) and for some bases of SU_6 and SU_3 (Bickerstaff et al. 1982, reproduced in Appendix). In Chapter IV, we show that algebraic formulae of 6j symbols for SU_n can be obtained using this method. These formulae appear as factored polynomials in n. Combined with the use of the composite labelling of irreps of the unitary groups, the 6j symbols display a new symmetry which originates from the transpose-conjugate symmetry in the symmetric groups. This symmetry is applied to Young diagrams and is due to the appearance of the one-dimensional alternating irrep $[1^f]$ of the symmetric group S_f. The manifestation of this transpose conjugate symmetry within the algebraic formulae of SU_n 6j symbols is essentially the replacement of n by -n. The unitary group modification rules, which arise from the determental properties of Schur functions and the many one-dimensional irreps of U_n, are also discussed in relation to the n-dependent formulae. Much work is required to fully understand the form of these algebraic formulae.

In the literature several other methods have been used to calculate 6j and 3jm symbols of the unitary group. These include projection operator construction of states involving a transformation from the known basis, for example the construction of the $SU_3 \supset SO_3$ basis from the $SU_3 \supset U_1 \times SU_2$ basis (Akiyama and Draayer 1973). The ladder operator techniques of angular momentum theory have been used by Baird and Biedenharn and many of their collaborators for U_n (see Biedenharn and Louck 1982). The explicit determination of all finite dimensional unitary irreps of the family of U_n is an essential task of their method. The procedure employs an embedding of U_{n-1} in U_n to achieve a recursive algebraic structure. However the disadvantage of this approach is that it becomes exceedingly cumbersome as n increases.

The final topic to be considered in this thesis is the close relationship that exists between the symmetric groups and the unitary groups - the Schur-Weyl duality. Schur and Weyl utilized the theory of the symmetric group as a technique for studying the unitary groups (and in general all Lie groups) (Weyl 1946). The most complete treatment for the representations is Littlewood's explicit construction of the representations by an extension of Frobenius' method for defining invariant matrices (Littlewood 1940). The duality goes further than that expressed by the character theory. Many powerful equalities between various transformation factors of the symmetric groups and those of the unitary groups have been established by Kramer (1967, 1968), Sullivan (1973, 1975a, 1976) and Chen (1981). One of the important results coming from these equalities was the derivation by Kramer and Seligman (1969a) of the Regge symmetries for the 6j symbols of SU_2 and for the 3jm symbols of $SU_2 \supset U_1$.

In Chapters V and VI, we further extend the Schur-Weyl duality results. The group-subgroup transformation theory is used to derive three factorization lemmas. These lemmas form the foundation to the duality and give rise to three transformation factors which we call symmetric group-unitary group (S_f-U_p) duality factors. The importance of these factors lies in the fact that they relate the phase and multiplicity freedoms within the GH transformation theory for the symmetric groups to similar freedoms within the GH transformation theory for the unitary groups. We show in Chapter VI that there is insufficient freedom to choose all the duality factors although many can be chosen unity. One of several important topics is the distinction between the unitary group U_n and the special unitary group SU_n. The duality relations of Kramer and Seligman, Sullivan and Chen are derived from our lemmas. These relations give extensions to the Regge symmetries.

The duality results have been used (Bickerstaff, Joshi, Butler, Haase 1982) in a multiquark calculation to find the fraction of the colour neutral (unconfined) component and the colour-charged (confined) component, present in any multiquark state. If there are N quark flavours the multiquark states can be classified according to the group-subgroup scheme $SU_{6N} \supset SU_{2N} \times SU_3$ where SU_3 carries the colour quantum numbers and SU_{2N} carries the spin and flavour quantum numbers. The fractions of the confined and unconfined components where shown by duality to be independent of the number of quark flavours.

A somewhat trivial but nevertheless important aspect is notation and terminology. We follow Butler (1981) as much as possible and use this as a guide when choosing and presenting notation and terminology for the various new transformation factors of the GH transformation theory. In much of this thesis we use the Dirac bra-ket notation $\langle u|v\rangle$ for the transformation factors. However with regard to the symmetries of these factors, that is the phase freedom, complex conjugation and transposition factors, we use the matrix notation $B^{i}{}_{j}$ where B will include irrep labels of the groups. One departure of this notation from that in common usage is that we employ $A(\lambda)^{i'}{}_{i}$ to denote a 2jm symbol $\begin{pmatrix}\lambda^* & \lambda \\ i' & i\end{pmatrix}$. We have also used recoupling and coupling factors instead of 6j and 3jm symbols in discussing symmetries, in particular the duality symmetry, and the calculational methods. The reason is that the 6j and 3jm symbols introduce additional phrase and dimension factors which detract from the symmetries presented and the analogies with other relations involving the subduction, resubduction, induction and reinduction factors.

Other notational aspects include the exclusive use of Greek letters, $\lambda,\eta,\kappa,\lambda,\mu,\nu,\ldots$ to label irreps of groups, upper case Latin letters G H K L M N to label groups and lower case Latin letters i j k l m n to label the basis vectors of an irrep space. Lower case Latin letters g h k l are used to denote elements of the groups G H K L respectively and p q r for coset and double coset elements. The letters a b c d r s denote product and branching multiplicity labels. Finally n p q are used to denote the rank of the unitary groups while f and t denote the different symmetric groups. Where the usage of notation overlaps the correct interpretation can be obtained from the context.

From the mathematician's point of view, the physicist is behaving like a beginner who will not take the step from "two oranges and two oranges is four oranges" to "two plus two equals four".

G.W. Mackey
Introduction to Biedenharn and Louck, 1982.

CHAPTER II

GROUP-SUBGROUP TRANSFORMATION THEORY

1. INTRODUCTION

From the works of Schur, Frobenius and Weyl, we have a number of theorems connecting the character theory of the symmetric and the unitary groups. This connection, known as the Schur-Weyl duality, goes further in that many equations can be established to relate various pairs of transformation coefficients, one from each group. In Chapters V and VI, we wish to develop and continue other authors' recent extensions of the Schur-Weyl duality. For this purpose, we require some new results in the theory of transformation coefficients for arbitrary compact groups, particularly in the area of induced representations. In so doing, we present in Section 2 a new perspective on coupling (or isoscalar) factors, recoupling coefficients, and Kramer's 6f symbols, showing that they are all specialized transformation factors. Section 3 introduces some double coset bases. Section 4 reviews the process of inducing from subgroup to group and introduces the induction coefficient. Two other types of transformation factors involving induced representations are given in Sections 5 and 6. Their relationships to the induction coefficients are also given.

2. REMARKS ON TRANSFORMATION COEFFICIENTS

For our purposes we take a representation of a (finite or compact continuous) group as a vector space (or representation space) V of finite dimension together with a set of linear operators g which map V onto itself and which obey the group properties. Because we restrict ourselves to finite dimension unitary spaces, sections 4, 5 and 6 are restricted to induction of finite groups only. The representation is an irreducible representation (irrep) if under the action of the group operators, there is no invariant proper subspace of the representation space. In general the representation space is reducible and may be decomposed into a direct sum of irrep spaces of G. We write

$$V = \bigoplus_{z\gamma} V_{z\gamma}$$

where $V_{z\gamma}$ is an irrep space that belongs to the equivalence class of representation spaces labelled by γ. The label z is called the parentage label and distinguishes equivalent irrep spaces of V.

We define a basis for $V_{z\gamma}$ as the set

$$\{|z\gamma(G)i\rangle \quad : \quad i=1\ldots|\gamma|\} \tag{II.2.1}$$

We shall use the shorthand notation

$$|z\gamma i\rangle \equiv |z\gamma(G)i\rangle \quad . \tag{II.2.2}$$

The action of the group operators on this set determines an irreducible matrix representation (matrix irrep). We have

$$g\cdot|z\gamma i\rangle = |z\gamma i'\rangle\langle z\gamma i'|g|z\gamma i\rangle \qquad \text{for all } g\varepsilon G. \tag{II.2.3}$$

(The summation convention, which we use throughout this thesis, is to sum on indices (Greek or Latin) that occur only once in the bra or raised in a matrix, and only once in a ket or lowered in a matrix). We note that for equivalent irrep spaces, $V_{z\gamma}$ and $V_{z'\gamma}$ say, the irrep matrices in (II.2.3) may be different even through their characters are identical. However, it is always possible to choose the bases of the equivalent irrep spaces so that the irrep matrices are identical for all equivalent irrep spaces (for this is a definition of equivalent). We shall call such a basis a G basis of V. The irrep matrices in this G basis must be independent of the parentage label z. We write

$$\langle z\gamma i|g|z'\gamma' i'\rangle = \delta^{z}{}_{z'}\delta^{\gamma}{}_{\gamma'}\ \gamma(g)^{i}{}_{i'} \qquad \text{(II.2.4)}$$

and shall refer to this condition as the matrix invariance requirement. An alternative G basis may be formed for V by taking linear combinations of the G basis vectors $|z\gamma i\rangle$,

$$|z'\gamma' i'\rangle = |z\gamma i\rangle\langle z\gamma i|z'\gamma' i'\rangle \quad . \qquad \text{(II.2.5)}$$

The irreducibility (Schur lemma 1) requires that the transformation be diagonal in γ. The two G bases may give rise to identical irrep matrices. In such an event the two G bases are said to be equivalent. By Schur lemma 2, the transformation between two equivalent G bases must be diagonal and independent of i

$$\langle z\gamma i|z'\gamma' i'\rangle = \langle z\gamma|z'\gamma\rangle\delta^{\gamma}{}_{\gamma'}\delta^{i}{}_{i'} \qquad \text{(II.2.6)}$$

The elements $\langle z\gamma|z'\gamma\rangle$ are elements of a unitary matrix . See for example Butler (1981 pp.15-18).

Let us now add more group structure in the form of a subgroup H of G. The irrep spaces of G are, in general, reducible representation spaces of H. If $\eta, \eta' \ldots$ denote the irreps of H, we write the decomposition of $V_{z\gamma}$ into these scripts as

$$V_{z\gamma} = \bigoplus_{a\eta} V_{z\gamma a\eta}$$

where $a = 1,2,\ldots|\gamma:\eta|$ is the branching multiplicity label which distinguishes the $|\gamma:\eta|$ occurrences of η in $V_{z\gamma}$. A basis of $V_{z\gamma}$ is given by the set

$$\{|z\gamma(G)a\ \eta(H)j\rangle \equiv |z\gamma a\eta j\rangle$$

$$: a = 1,\ldots|\gamma:\eta|,\ \eta(H), j=1,\ldots,|\eta|\} \qquad \text{(II.2.7)}$$

The group operator action on this set is

$$g\cdot|z\gamma a\eta j\rangle = |z\gamma a'\eta' j'\rangle\langle\gamma a'\eta' j'|g|\gamma a\eta j \quad \text{for } g\varepsilon G \qquad \text{(II.2.8a)}$$

$$= |z\gamma a\eta j'\rangle\langle\gamma a\eta j'|g|\gamma a\eta j\rangle \qquad \text{for } g\varepsilon H \qquad \text{(II.2.8b)}$$

Note that if H is the identity group E with the single irrep O(E) then $|\gamma:\dot{0}| = |\gamma|$ and $|z\gamma i00\rangle = |z\gamma i\rangle$ where i has replaced a.

In a similar manner to the definition of a G basis, we define a GH basis as a basis satisfying the matrix invariance requirement for both G and H while an H basis would be obtained by retaining it for H but relaxing it for G. Thus for GH and H bases one has for $h \varepsilon H$

$$<\gamma a'\eta'j'|h|\gamma a\eta j = \delta^{a'}{}_{a}\delta^{\eta'}{}_{\eta}\ \eta(h)^{j'}{}_{j} \qquad \text{(II.2.9)}$$

In the remainder of this section we wish to ignore transformations in the parentage label of G bases of GH bases expressed by (2.6). Instead, we look at the properties of the transformation between inequivalent GH bases but equivalent H bases. The transformation may be written

$$|z\gamma'a'\eta'j'> = |z\gamma a\eta j><\gamma a\eta j|\gamma'a'\eta'j'> \qquad \text{(II.2.10)}$$

Irreducibility (Schur lemma 1 applied to both G and H) forces $\gamma'=\gamma$ and $\eta'=\eta$, and Schur lemma 2 applied to H forces $j'=j$. Hence

$$<\gamma a\eta j|\gamma'a'\eta'j'> = <\gamma a\eta|\gamma a'\eta>\delta^{\gamma}{}_{\gamma'}\delta^{\eta}{}_{\eta'}\delta^{j}{}_{j'} \qquad \text{(II.2.11)}$$

See Bickerstaff (1980). The property (2.11) of this transformation, is of great importance in the study of the Racah-Wigner algebra. We call the association transformation coefficient a GH transformation factor to emphasise its independence from the subgroup basis labels. With each such factored transformation, we associate the diagram of figure 2.1. The alternative routes correspond to different GH basis kets, which have the same H basis. The transformation

Figure II.2.1

$$\begin{tikzcd}
{} \arrow[d, "z"] \\
\gamma(G) \arrow[d, bend right=60, "a"'] \arrow[d, bend left=60, "a'"] \\
\eta(H) \arrow[d, "j"] \\
0(E)
\end{tikzcd}$$

factor $\langle\gamma a'\eta|\gamma a\eta\rangle$ is the archetype of a class of transformation factors for any two group-subgroup schemes with common group and subgroup. For example, the transformation factor for the schemes of figure 2.2 is what we shall call the coupling factor and denote as

$$\langle\kappa\mu b\gamma a\eta|(\kappa a_1\lambda;\mu a_2\nu)c\eta\rangle$$

$$\equiv \left\langle \begin{array}{c|cc} \kappa\mu & \kappa & \mu \\ b & a_1 & a_2 \\ \gamma & \lambda & \nu \\ a & \multicolumn{2}{c}{c} \\ \eta & \multicolumn{2}{c}{\eta} \end{array} \right\rangle \qquad \text{(II.2.12)}$$

Figure II.2.2

$$\begin{tikzcd}
 & {} \arrow[d, "xy"] & \\
 & \kappa\mu(K\times M) \arrow[dl, "b"'] \arrow[dr, "a_1a_2"] & \\
\gamma(G) \arrow[dr, "a"'] & & \lambda\nu(L\times N) \arrow[dl, "c"] \\
 & \eta(H) \arrow[d, "j"] & \\
 & 0(E) &
\end{tikzcd}$$

We observe that if L and N are the identity group, E, then, (2.12) reduces to a coupling coefficient

$$\langle\kappa\mu b\gamma i|(\kappa k0,\mu m0)00\rangle\equiv\langle\kappa\mu b\gamma i|\kappa k\mu m\rangle \qquad \text{(II.2.13)}$$

where a, a_1 and a_2 are replaced by i, k and m respectively. The special branchings, $K\times M\supset G$ and $L\times N\supset H$, where K and M are isomorphic to G, and where L and N are isomorphic to H, are termed couplings in the literature on the Racah-Wigner algebra, and hence our use of the terms coupling factor and coupling coefficient.

The schemes of figure 2.3 give another special transformation factor which we shall call the recoupling factor

$$\langle(\lambda\mu)a\eta,\nu,b\gamma|\lambda(\mu\nu)c\kappa,d\gamma\rangle\quad . \qquad \text{(II.2.14)}$$

Figure II.2.3

$$\begin{tikzcd}
 & \lambda\mu\nu\,(L\times M\times N) \arrow[dl, "a"'] \arrow[dr, "c"] & \\
\eta\nu\,(H\times N) \arrow[dr, "b"'] & & \lambda\kappa\,(L\times K) \arrow[dl, "d"] \\
 & \gamma\,(G) &
\end{tikzcd}$$

These factors can be defined with reference to coupling coefficients of (2.13),

$$\begin{aligned}
&\langle(\lambda\mu)a\eta,\nu,b\gamma|\lambda(\mu\nu)c\kappa,d\gamma\rangle \\
&\quad\times\ \langle\lambda\kappa d\gamma i|\lambda\ell\kappa k\rangle\langle\mu\nu c\kappa k|\mu m\nu n\rangle \qquad \text{(II.2.15)} \\
&\quad=\ \langle\eta\nu b\gamma i|\eta j\nu n\rangle\langle\lambda\mu a\eta j|\lambda\ell\mu m\rangle \quad .
\end{aligned}$$

If all six groups G,H,K,L,M and N, are isomorphic, then (2.14) is the recoupling coefficient, which is well known in the Racah-Wigner algebra and which is usually defined by (2.15). See Butler (1981, eqn. 3.2.17). To be consistent with our terminology, we call this recoupling coefficient a recoupling factor.

A further example is the factor associated with figure 2.4 which we write as

$$\langle\gamma a\eta(b\lambda\mu)\nu|\gamma c\lambda\kappa(d\mu\nu)\rangle \qquad \text{(II.2.16)}$$

Figure II.2.4

$$\begin{tikzcd}[column sep=small]
 & \gamma(G) \arrow[dl] \arrow[dr] & \\
\eta\nu(H\times N) \arrow[dr] & & \lambda\kappa(L\times K) \arrow[dl] \\
 & \lambda\mu\nu(L\times M\times N) &
\end{tikzcd}$$

Kaplan (1962a,b) and Horie (1964) introduced such transformation factors for chains of symmetric groups. Kramer (1967) has analysed and calculated these factors, which he termed 6f symbols, for all cases without multiplicity. We shall use the term resubduction factor for any transformation of the form given by figure 2.4.

The resubduction factor, in analogy with the recoupling factor, can be defined by four transformation coefficients of the type

$$\langle \gamma a \eta j \kappa k | \gamma i \rangle \quad , \tag{II.2.17}$$

which describe the decomposition of irreps of G according to the group-subgroup chain $G \supset H \times K$. Such coefficients will be called subduction coefficients. They have been discussed, in connection with the symmetric groups, by several authors (see Kramer 1967, 1968). The relationship with the resubduction factor is given as

$$\begin{aligned}
&\langle \gamma a \eta (b \lambda \mu) \nu | \gamma c \lambda \kappa (d \mu \nu) \rangle \\
&\quad \times \langle \kappa d \mu m \nu n | \kappa k \rangle \langle \gamma a \eta j \nu n | \gamma i \rangle \\
&\quad = \langle \eta b \lambda \ell \mu m | \eta j \rangle \langle \gamma c \lambda \ell \kappa k | \gamma i \rangle \quad .
\end{aligned} \tag{II.2.18}$$

To continue the analogy with coupling theory, and for completeness, we define the subduction factor

$$\langle \gamma c \kappa (a_1 \mu) \lambda (a_2 \nu) | \gamma a \eta b \mu \nu \rangle \qquad \text{(II.2.19)}$$

which describes the transformation between the group-subgroup scheme given in figure 2.5. When H is the identity group, (2.19) reduces to a subduction coefficient.

$$\langle \gamma c \kappa (k0) \lambda (\ell 0) | \gamma i0000 \rangle = \langle \gamma c \kappa k \lambda \ell | \gamma i \rangle \qquad \text{(II.2.20)}$$

where a is replaced by i, a_1 by k, and a_2 by ℓ.

Figure II.2.5

$$\begin{tikzcd}
 & \gamma(G) \arrow[dl, "c"'] \arrow[dr, "a"] & \\
\kappa\lambda(K\times L) \arrow[dr, "a_1 a_2"'] & & \eta(H) \arrow[dl, "b"] \\
 & \mu\nu(M\times N) &
\end{tikzcd}$$

3. DOUBLE COSET BASES

This section introduces the double coset decomposition, which has been studied extensively by Sullivan (1980, and references therein). In contrast to the previous section, we study bases of $V_{z\gamma}$ for chains involving G and $G_g \equiv gGg^{-1}$ where $g \varepsilon G$. Clearly, for any subgroup M of G, $M_g \equiv gMg^{-1} = \{gmg^{-1} : m \varepsilon G\}$ is isomorphic to M. We write $M \overset{g}{\simeq} M_g$. The case for which g is a double coset representative is of particular interest.

The set $H\backslash G/K$ of double cosets HqK of a group G, with respect to two subgroups H and K, is obtained by writing each element $g \in G$ as

$$g = hqk \quad \text{where} \quad h \in H \quad \text{and} \quad k \in K \quad . \qquad \text{(II.3.1)}$$

The elements q are called double coset representatives (see Coleman 1966, Bradley and Cracknell 1972). For each q, we have isomorphic subgroups $L(q)$, $L_q(q)$, $L_{q^{-1}}(q)$, defined by

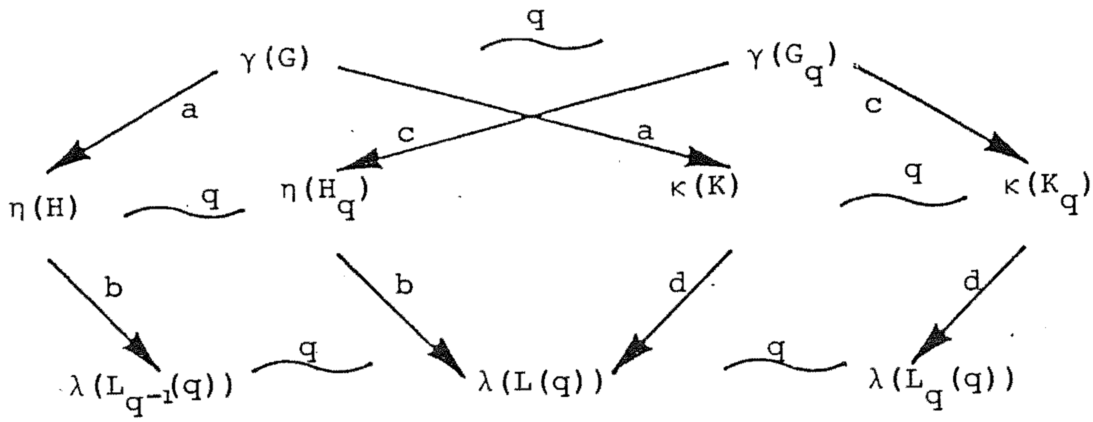
$$L(q) \equiv H_q \cap K, \; L_q(q) \equiv qL(q)q^{-1}, \; L_{q^{-1}}(q) \equiv q^{-1}L(q)q \quad . \qquad \text{(II.3.2)}$$

A space, $V_{z\gamma}$, which is an irrep space of G, is simultaneously an irrep space of G_g, for any $g \in G$. But a G basis is not simultaneously a G_g basis, since Schur's lemmas would require equality of the pair of irrep matrices $\gamma(g')$ and $\gamma(gg'g^{-1})$ for all $g' \in G$. Therefore, we consider transformations between the bases represented by the chains, $G \supset K \supset L$, $G \supset H \supset L_{q^{-1}}$, $G_q \supset H_q \supset L$ and $G_q \supset K_q \supset L_q$, as shown in figure 3.1. (For simplicity, we shorten $L(q)$ to L, but note the dependency of L on q). The action of q is to take a GKL basis vector (respectively a $GHL_{q^{-1}}$ basis vector) into a $G_qK_qL_q$ basis vector (respectively a G_qH_qL basis vector). That is,

$$q\cdot|z\gamma(G)c\kappa(K)d\lambda(L)\ell\rangle = |z\gamma(G_q)c\kappa(K_q)d\lambda(L_q)\ell\rangle \quad , \text{(II.3.3)}$$

$$q\cdot|z\gamma(G)a\eta(H)b\eta(L_{q^{-1}})\ell\rangle = |z\gamma(G_q)a\eta(H_q)b\lambda(L)\ell\rangle . \text{(II.3.4)}$$

Figure II.3.1



The overlap between the basis vectors of (3.3) and those of the GHL_{q-1} basis defines Sullivan's (1973) double coset matrix elements (DCMEs), which clearly have the factorization property (cf. 2.11).

$$\langle\gamma(G)a\eta(H)b\lambda'(L_{q-1})\ell'|q|\gamma(G)c\kappa(K)d\lambda(L)\ell\rangle$$

$$= \langle\gamma(G_q)a\eta(H_q)b\lambda'(L)\ell'|\gamma(G)c\kappa(K)d\lambda(L)\ell\rangle$$

$$= \langle\gamma(G\ \ )a\eta(H\ \ )b\lambda(L)|\gamma(G)c\kappa(K)d\lambda(L)\rangle\ \delta^{\lambda'}_{\lambda}\ \delta^{\ell'}_{\ell}\ . \quad \text{(II.3.5)}$$

This type of transformation factor has been considered by Reid and Butler (1980, 1982), in their discussion of different (rotated) point group embeddings, such as $O\supset D_4\supset C_2$ and $O\supset D_3\supset C_2$ (see also Butler, 1981 §5.3). The resubduction factor of (2.16) can be seen as a special case of a DCME, for which $G_q\rightarrow G$, $H_q\rightarrow H\times N$, $K\rightarrow L\times K$, $L\rightarrow L\times M\times N$ and a choice of bases such that q=e, the identity element. It is not until Mackey's subgroup theorem is introduced in Section 6 that we shall be able to use a powerful completeness relation over the series of subgroups, $L_q(q)$.

4. BASES OF INDUCED SPACES

The preceding sections discussed transformations arising from the concept of the decomposition of an irrep space, $V_{z\gamma}$, of a group G into irreps of a subgroup, H. A second concept is that of induction, in which a representation space of G is obtained from an irrep space, $V_{y\eta}$, of a subgroup, H. Induction takes the tensor product of $V_{y\eta}$ with the left coset space, $V_{H\backslash G}$. This induced representation space, $V_{H\backslash G} \otimes V_{y\eta} = V_{y\eta(H)\uparrow G}$ (or written simply as $V_{y\eta\uparrow}$) has basis bectors

$$|y\eta\uparrow pj\rangle \equiv |y\eta(H_p)j\rangle \equiv |p\rangle|y\eta(H)j\rangle \ , \qquad \text{(II.4.1)}$$

where $H_p = pHp^{-1}$.

The basis $\{|p\rangle\}$ of $V_{H\backslash G}$, like the basis $\{|y\eta j\rangle\}$, is not unique, but it is important to note that once chosen both bases remain fixed. The following results which we need are contained in Coleman (1966) and Bradley and Cracknell (1972).

The action of $g \in G$ on $|y\eta\uparrow pj\rangle$ is given as

$$g\cdot|y\eta\uparrow pj\rangle = |y\eta\uparrow p'j'\rangle \ \eta\uparrow(g)^{p'j'}{}_{pj} \qquad \text{(II.4.2)}$$

$$= |y\eta\uparrow p'j'\rangle \ \delta^{p'}{}_{p_o}\,\eta(p_o{}^{-1}gp)^{j'}{}_{j} \qquad \text{(II.4.3)}$$

where p_o is that unique representative for which $p_o{}^{-1}gp \in H$. Observe that $\{|y\eta\uparrow pj\rangle\}$ is not a G basis but is a basis of a reducible representation space of G of dimension $|\eta|\times|G|/|H|$. Its transformation to a G basis,

$$|\gamma\eta\uparrow pj\rangle = |\gamma\eta\uparrow a\gamma i\rangle\langle\eta\uparrow a\gamma i|\eta\uparrow pj\rangle \quad , \qquad \text{(II.4.4)}$$

gives rise to matrix elements labelled by the index sets pj and $a\gamma i$. We shall call the elements, $\langle\eta\uparrow a\gamma i|\eta\uparrow pj\rangle$, induction coefficients. Their factorization will be given in Section 6.

The Frobenius reciprocity theorem states that the number of occurrences of γ in $\eta(H)\uparrow G$ equals the number of occurrences of the space $\eta(H)$ in $\gamma(G)$, that is $|\eta\uparrow:\gamma| = |\gamma:\eta|$. This implies that $a=1,\ldots,|\gamma:\eta|$ in (II.4.4). In the case when $\eta(H)$ is the identity irrep $0(E)$, the induced space $0(E)\uparrow G$ is the $|G|$-dimensional regular representation space of G. Then, the reciprocity theorem implies that if γ is an irrep of G, $|0\uparrow:\gamma| = |\gamma:0|$, and hence $\sum_{\gamma}|\gamma|^2 = |G|$.

5. REINDUCTION FACTORS

In this section we define the reinduction factor and show its relationship to the induction coefficients. A special type of reinduction factor, which appears in later sections, is also given.

Consider the chain $G\supset H\supset L$. The induction of an irrep $\lambda(L)$ in G is equivalent to, or gives the same space as, the two step process of inducing first into H, then into G. We have

$$\lambda(L)\uparrow G \simeq (\lambda(L)\uparrow H)\uparrow G \qquad \text{(II.5.1)}$$

(Coleman 1966, thm. 4). Hence, given the chain $G\supset K\supset L$, (see figure 5.1)

$$(\lambda(L)\uparrow K)\uparrow G \simeq (\lambda(L)\uparrow H)\uparrow G\ . \qquad \text{(II.5.2)}$$

The transformation between the bases, obtained by (i) induction to the intermediate group (H or K), (ii) decomposition to its irreps (η or κ), (iii) further induction to G, and (iv) decomposition to irreps of G, gives rise to the transformation factor

$$\langle\lambda(L)\uparrow G\ a\ \eta(H)\uparrow G\ b\ \gamma(g)\,|\,\lambda(L)\uparrow G\ c\ \kappa(K)\uparrow\ d\ \gamma(G)\rangle$$

$$\equiv \langle\lambda\uparrow\ a\ \eta\uparrow\ b\ \gamma\,|\,\lambda\uparrow\ c\ \kappa\uparrow\ d\ \gamma\rangle\ . \qquad \text{(II.5.3)}$$

We call this factor a reinduction factor. In its definition, we have used the conditions of Schur's lemmas to omit any

Figure II.5.1

$$\begin{tikzcd}
 & \lambda(L)\uparrow G \arrow[dl, "a"'] \arrow[dr, "c"] & \\
\eta(H)\uparrow G \arrow[dr, "b"'] & & \kappa(K)\uparrow G \arrow[dl, "d"] \\
 & \gamma(G) &
\end{tikzcd}$$

parentage label for $\lambda(L)$ and the basis index i for $\gamma(G)$. On a point of notation, observe that the arrows in figure 5.1 are downward, as in the figures of Section 2, for the reason that the (reducible) representation space $\lambda(L)\uparrow G$ may be written as a direct sum of induced representation spaces of either $H\uparrow G$ or $K\uparrow G$, each of which is a direct sum of representation spaces V_γ of G.

The reinduction factor is unitary over the index sets $a\eta b$ and $c\kappa d$. As a result of the Frobenius reciprocity theorem, the transformation factor (see figure 5.2)

$$\langle\gamma(G)\ a\ \eta(H)\ b\ \lambda(L)|\gamma(G)\ c\ \kappa(K)\ d\ \gamma(L)\rangle \quad \text{(II.5.4)}$$

is, for fixed γ and λ, unitary over the same index sets. It is possible to make phase and multiplicity choices such that, for the same labels, the two factors are equal.

Figure II.5.2

$$\begin{tikzcd}
& \gamma(G) \arrow[dl, "a"'] \arrow[dr, "c"] & \\
\eta(H) \arrow[dr, "b"'] & & \kappa(K) \arrow[dl, "d"] \\
& \lambda(L) &
\end{tikzcd}$$

The reinduction factor can be defined in terms of four induction factors. This is shown by giving alternative bases for the induced space, $\lambda(L)\uparrow G$, and using the fact that each element of G can be written

$$g = p_1 h = p_1 p_2 \ell' = p_3 k = p_3 p_4 \ell' = p\ell',$$

where $h\varepsilon H$, $k\varepsilon K$ and $\ell'\varepsilon L$ with $p_1\varepsilon G/H$, $p_2\varepsilon H/L$, $p_3\varepsilon G/K$ and $p_4\varepsilon K/L$. Thus

$$\begin{aligned}
|\lambda\uparrow p\ell\rangle &= |\lambda\uparrow p_1 p_2 \ell\rangle \\
&= |\lambda\uparrow p_1 a\eta j\rangle\langle\lambda\uparrow a\eta j|\lambda\uparrow p_2\ell\rangle \qquad \text{(II.5.5)} \\
&= |\lambda\uparrow a\eta\uparrow b\gamma i\rangle\langle\eta\uparrow b\gamma i|\eta\uparrow p_1 j\rangle\langle\lambda\uparrow a\eta j|\lambda\uparrow p_2\ell\rangle
\end{aligned}$$

and similarly,

$$\begin{aligned}
|\lambda\uparrow p\ell\rangle &= |\lambda\uparrow p_3 p_4 \ell\rangle \\
&= |\lambda\uparrow p_3 c\kappa k\rangle\langle\lambda\uparrow c\kappa k|\lambda\uparrow p_4\ell\rangle \qquad \text{(II.5.6)} \\
&= |\lambda\uparrow c\kappa\uparrow d\gamma i\rangle\langle\kappa\uparrow d\gamma i|\kappa\uparrow p_3 k\rangle\langle\lambda\uparrow c\kappa k|\lambda\uparrow p_4\ell\rangle \quad .
\end{aligned}$$

The overlap of these two equations gives the desired result,

$$\langle \lambda{\uparrow}a\eta{\uparrow}b\gamma | \lambda{\uparrow}c\kappa{\uparrow}d\gamma\rangle\langle\kappa{\uparrow}d\gamma i|\kappa{\uparrow}p_3 k\rangle\langle\lambda{\uparrow}c\kappa k|\lambda{\uparrow}p_4 \ell\rangle$$
$$= \langle\eta{\uparrow}b\gamma i|\eta{\uparrow}p_1 j\rangle\langle\lambda{\uparrow}a\eta j|\lambda{\uparrow}p_2\ell\rangle \quad . \qquad \text{(II.5.7)}$$

The analogy with the recoupling and resubduction factors, (2.14, 2.17), which may be defined with respect to four coupling and subduction coefficients respectively, is the reason for our choice of name, reinduction factor.

One particular type of reinduction factor which we shall be requiring in later sections, is the factor corresponding to figure 5.3. This will be written as

$$\langle(\lambda\mu){\uparrow}a\eta,\nu,{\uparrow}b\gamma|\lambda(\mu\nu){\uparrow}c\kappa,{\uparrow}d\gamma\rangle \qquad \text{(II.5.8)}$$

which, by the Frobenius reciprocity theorem, has the same unitary properties as the resubduction factor of (2.16) and figure 2.4. For identical irrep and multiplicity labels, we can equate the reinduction factor with the resubduction factor. This reinduction factor can also be written as four induction coefficients (cf. 5.7),

$$\langle(\lambda\mu){\uparrow}a\eta,\nu,{\uparrow}b\gamma|\lambda(\mu\nu){\uparrow}c\kappa,{\uparrow}d\gamma\rangle$$
$$\times\ \langle\lambda\kappa{\uparrow}d\gamma i|\lambda\kappa{\uparrow}p_3\ell k\rangle\langle\mu\nu{\uparrow}c\kappa k|\mu\nu{\uparrow}p_4 mn\rangle$$
$$= \langle\eta\nu{\uparrow}b\gamma i|\eta\nu{\uparrow}p_1 jn\rangle\langle(\lambda\mu){\uparrow}a\eta j|\lambda\mu{\uparrow}p_2\ell m\rangle \quad . \qquad \text{(II.5.9)}$$

<u>Figure II.5.3</u>

$$\begin{tikzcd}
 & \lambda\mu\nu(L\times M\times N){\uparrow}G \arrow[dl, "a"'] \arrow[dr, "c"] & \\
\eta\nu(H\times N){\uparrow}G \arrow[dr, "b"'] & & \lambda\kappa(L\times K){\uparrow}G \arrow[dl, "d"] \\
 & \gamma(G) &
\end{tikzcd}$$

Observe that each induction coefficient G induces an irrep of a direct product subgroup into a single group, for example, from figure 5.3, LxM is induced to H.

6. THE INDUCTION FACTOR AND MACKEY'S SUBGROUP THEOREM

We can form two special bases for the induced representation space $\eta(H)\uparrow G$. The first is the GK basis labelled as

$$\{|\eta\uparrow a\gamma b\kappa k\rangle \ :\ a=1,\dots,|\gamma:\eta|, \gamma(G),\ b=1,\dots,|\gamma:\kappa|,\ \kappa(K),\ k=1,\dots,|\kappa|\} \qquad \text{(II.6.1)}$$

where, as before, the Frobenius reciprocity theorem gives the range of a. The second basis is obtained by writing each coset representative, p of $H\backslash G$, as p=rq where $q\varepsilon H\backslash G/K$ and $r\ \varepsilon\ L(q)\backslash K$ (see Bradley and Cracknell 1972, thm. 4.7.5), and choosing the $HL_{q^{-1}}(q)$ basis for the space V_η. By writing the H basis in this q-dependent fashion, the basis vectors of $\eta(H)\uparrow G$ may be written

$$\begin{aligned} |\eta\uparrow pj\rangle &\equiv |p\rangle|\eta(H)j\rangle \\ &\equiv |rq\rangle|\eta(H)\ c\ \lambda(L_{q^{-1}})\ell\rangle\langle\eta c\lambda\ell|\eta j\rangle \\ &\equiv |r\rangle|\eta(H_q)c\ \lambda(L)\ell\rangle\langle\eta c\lambda\ell|\eta j\rangle \quad , \end{aligned} \qquad \text{(II.6.2)}$$

where the q dependence of L must be remembered, and where (3.3) has been used, namely,

$$q\cdot|\eta(H)c\ \lambda(L_{q^{-1}})\ell\rangle = |\eta(H_q)c\ \lambda(L)\ell\rangle \quad . \qquad \text{(II.6.3)}$$

For each $qc\lambda$, the basis vectors

$$\{|r\rangle|\eta(H_q)c\ \lambda(L)\ell\rangle \ :\ r\ \varepsilon\ L\backslash K,\ \ell=1,\dots,|\lambda|\} \qquad \text{(II.6.4)}$$

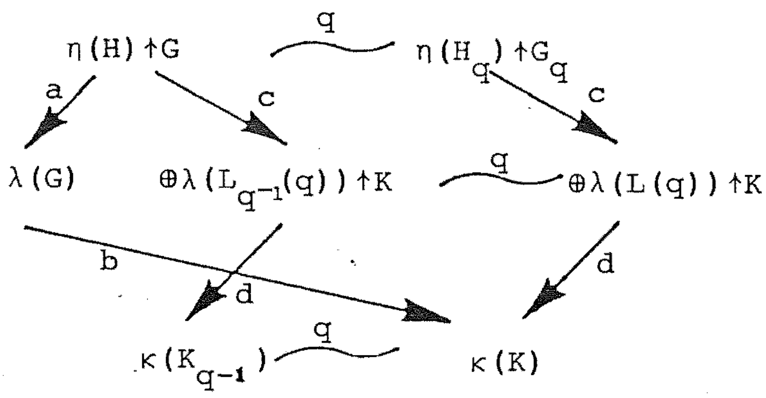
form an induced space, $\lambda(L)\uparrow K$, for which a K basis may be chosen. By this route, we have chosen our second basis for the original $\eta(H)\uparrow G$ space. It is

$$\{|\eta(H_q)c\ \lambda(L(q))\uparrow d\ \kappa(K)k\rangle \equiv |q\eta c\lambda\uparrow d\kappa k\rangle :$$
$$q \in H\backslash G/K \quad c=1,\dots,|\eta:\lambda| \ , \quad \lambda(L(q)),$$
$$d=1,\dots,|\lambda:\kappa| \ , \ \kappa(K) \ , \ k=1,\dots,|\kappa|\} \quad . \quad \text{(II.6.5)}$$

The overlap between the basis vectors of (6.1) and (6.5) defines a new transformation factor (see figure 6.1), which we shall call the induction factor,

$$\langle\eta(H_q)c\ \lambda(L(q))\uparrow d\ \kappa(K)|\eta(H)\uparrow a\ \gamma(G)\ b\ \kappa(K)\rangle$$
$$\equiv \langle q\eta c\lambda\uparrow d\kappa|\eta\uparrow a\gamma b\kappa\rangle \qquad \text{(II.6.6)}$$

Figure II.6.1



It is to be emphasized that the bra belongs to $\eta(H)\uparrow G$ through the action of q, which has been absorbed into the bra by (6.3). The factor is unitary on the index sets $a\gamma b$ and $c\ \lambda(L(q))d$, where the summation includes all possible subgroups, $L(q)$, one for each coset representative q. Thus, the induction factor differs from the transformation factors of the earlier sections, in that the summation on c $\lambda(L(q))$d involves groups, as well as irreps and multiplicities. In the language of Mackey's subgroup theorem (Coleman 1966, thm. 8, Bradley and Cracknell 1972, thm. 4.7.6), the induction factor transforms between the representations

$$[\eta(H)\uparrow G]\downarrow K \quad \text{and} \quad \bigoplus_q [\eta(H_q)\downarrow L(q)]\uparrow K \quad . \qquad \text{(II.6.7)}$$

We remark that, for the case in which K is the identity group, the induction factor is just the induction coefficient defined in (4.4). In (6.6), the multiplicity labels, b and c, label a G basis for $\gamma(G)$ and a H basis for $\eta(H)$, respectively. Since p=eq, each double coset is a coset of H\G. Also, L(q)=E for all q since K⊃L(q). Thus, (6.6) is rewritten

$$\langle \eta(H)\uparrow a\ \gamma(G)i\ \ 0(E)\,|\,\eta(H_q)j\ \ 0(E)\uparrow o\ \ 0(E)\rangle \qquad \text{(II.6.8)}$$

$$\doteq \langle \eta\uparrow a\,\gamma i\,|\,\eta\uparrow qj\rangle \quad ,$$

where we have used (4.1), and replaced b by i and c by j. Comparing (6.8) with (2.13) and (2.20) the choice of name for this transformation factor becomes apparent.

In an alternative approach to induction theory using basis projection operators (Young symmetrizers), Sullivan (1975) has shown that the weighted double coset matrix element (WDCME),

$$\left[\frac{|\gamma|\,|\lambda|\,|H|\,|K|}{|G|\,|L|\,|\eta|\,|\kappa|}\right]^{\frac{1}{2}} \langle \gamma(G)a\ \ \eta(H)c\ \ \lambda(L_{q^{-1}})|q|\gamma(G)b\ \ \kappa(K)d\ \ \lambda(L)\rangle \ , \qquad \text{(II.6.9)}$$

describes the transformation between the two basis schemes, (6.1) and (6.5), of the induced representation space. The relationship between the WDCME and our induction factor involves definite phase and multiplicity choices, and is not established here. However, with such a choice, we observe that the induction factor must satisfy new unitary conditions. Since the DCME is unitary on the index sets $a\eta c$ and $b\kappa d$, the induction factor must be unitary over the same index set. Hence,

$$\sum_{b\kappa d} \left[\frac{|\gamma||\lambda||H||K|}{|G||L||\eta||\kappa|}\right] \langle \eta' \uparrow a'\gamma b\kappa | q\eta' c'\lambda\uparrow d\kappa\rangle . \langle q\eta c\lambda\uparrow d\kappa | \eta\uparrow a\gamma b\kappa\rangle = \delta^{a'}_{a}\ \delta^{\eta'}_{\eta}\ \delta^{c'}_{c} \quad , \qquad \text{(II.6.9a)}$$

$$\sum_{a\eta c} \left[\frac{|\gamma||\lambda||H||K|}{|G||L||\eta||\kappa|}\right] \langle q\eta c\eta\uparrow d'\kappa' | \eta\uparrow a\gamma b'\kappa\uparrow\rangle . \langle \eta\uparrow a\gamma b\kappa | q\eta c\lambda\uparrow d\kappa\rangle = \delta^{b'}_{b}\delta^{\kappa'}_{\kappa}\ \delta^{d'}_{d} \quad , \qquad \text{(II.6.9b)}$$

where γ and λ are fixed.

As an example of the induction schemes given by figure 6.1, we take the symmetric groups, $H=S_3 x S_2$, $G=S_5$, $K=S_4 x S_1$, and the induced space $\mu\uparrow=21x2\uparrow$. (We use partitions of ℓ into p parts, $(\lambda_1\lambda_2\ldots\lambda_p)$, to label the irreps of S_ℓ. See Littlewood (1940) or Wybourne (1970)). The coset space, $H\backslash G$, whose representatives p are assumed chosen, has dimension 10, so that $|\mu\uparrow|=10\times2\times1=20$. The Frobenius reciprocity theorem gives the decomposition of the induced space into G. We have

$$21\times2(H)\uparrow G \supset 41 + 32 + 31^2 + 2^21 \quad (G) \ . \qquad \text{(II.6.10)}$$

The GK decomposition follows simply:

$$41(G) \supset (4+31) \times 1 \quad (K)$$
$$32(G) \supset (31+21^2) \times 1 \quad (K)$$
$$31^2(G) \supset (31+21^2) \times 1 \quad (K) \qquad \text{(II.6.11)}$$
$$2^21(G) \supset (2^2+21^2) \times 1 \quad (K) \quad .$$

The alternative basis is obtained by first choosing the double cosets, of which there are two. Selecting $q_1=(e)$ and $q_2=(35)$, the subgroups, $L(q)$, of K are respectively

$$L(q_1)=S_3\times S_1\times S_0\times S_1 \quad \text{and } L(q_2)=S_1\times S_1\times S_1\times S_0 \quad . \qquad \text{(II.6.12)}$$

The identity group, S_0, is inserted for convenience. The decomposition of $21\times1(H_q)$ into each $L(q)$ is given as

$$21\times2(H_{q_1}) \supset 21\times1\times0\times1 \quad (L(q_1)), \qquad \text{(II.6.13)}$$

and

$$21\times2(H_{q_1}) \supset 2\times2\times1\times0 + 1^2\times2\times1\times0 \quad (L(q_2)) \quad . \qquad \text{(II.6.14)}$$

The coset spaces $V_{L(q)\backslash K}$ are determined by the choices of double cosets, q, and cosets, p, but here we do not need to specify them. We remark that each $V_{L(q)\backslash K}$ is a product of two coset spaces, since we are performing two inductions - one into S_4 and the other into S_1. The dimensions of the coset spaces are

$$|L(q_1)\backslash K| = 4\times1 \quad \text{and} \quad |L(q_2)\backslash K| = 6\times1. \qquad \text{(II.6.15)}$$

Note that $\sum_q |L(q)\backslash K| = |H\backslash G|$, which follows from the decomposition p=rq.

The resulting decomposition of each induced space, $\lambda(L(q))\uparrow K$, is again found by the Frobenius reciprocity theorem. We have

$$
\begin{array}{llll}
21\times1\times0\times1 & (L(q_1)\uparrow K) \supset (31+2^2+21^2)\times1 & (K) & \\
2\times2\times1\times0 & (L(q_2)\uparrow K) \supset (4+31+2^2)\times1 & (K) & \quad (II.6.16) \\
1^2\times2\times1\times0 & (L(q_2)\uparrow K) \supset (31+21^2)\times1 & (K) &
\end{array}
$$

The direct sum of all the irrep spaces of K is just the composition given in (6.11).

The unitary property of the induction factor is determined by choosing an irrep of K, for example $\kappa=31\times1$, and summing over the index sets which contain κ. In this example they are found to be

$$(a\gamma b) = (\gamma) = (41),\ (32),\ (31^2) \quad ,$$

and

$$(c\lambda d) = (\lambda) = (21\times1\times0\times1),\ (2\times2\times1\times0),\ (1^2\times2\times1\times0), \qquad (II.6.17)$$

where the multiplicity labels, a,b,c,d, can be omitted.

In later chapters, we will meet induction factors of similar type to that given in the above example. The induction schemes are outlined in figure 6.2, where we observe that we are inducing from a direct product group. The induction factor is written

$$
\begin{aligned}
&\langle q(\mu a_1\lambda;\nu a_2\rho)\uparrow c\eta \mid \mu\nu\uparrow a\gamma b\eta\rangle \qquad\qquad (II.6.18)\\
&\equiv \left\langle \begin{array}{c|c} q\mu\nu & \mu\nu\uparrow \\ a_1a_2 & a \\ \rho\sigma\uparrow & \gamma \\ c & b \\ \eta & \eta \end{array} \right\rangle
\end{aligned}
$$

Figure II.6.2

$$\begin{tikzcd}
 & \mu\nu(M\times N)\uparrow G \arrow[dl] \arrow[dr] \arrow[rr, phantom, "\overset{q}{\sim}"] & & \mu\nu(M_q\times N_q)\uparrow G_q \arrow[dr] & \\
\gamma(G) \arrow[drrr] & & \oplus\,\rho\sigma(R_{q-1}\times S_{q-1}(q))\uparrow H_{q-1} \arrow[dl] \arrow[rr, phantom, "\overset{q}{\sim}"] & & \oplus\rho\sigma(R\times S(q))\uparrow H \arrow[dl] \\
 & \eta(H_{q-1}) \arrow[rr, phantom, "\overset{q}{\sim}"] & & \eta(H) &
\end{tikzcd}$$

CHAPTER III

SYMMETRIES OF GH TRANSFORMATION FACTORS

1. INTRODUCTION

In the last chapter we defined the archetypal GH transformation factor and derived, as a special case, various factors for coupling, subduction, and induction schemes. For each scheme, we gave the relationship between the transformation coefficient and the transformation and "re"-transformation factors. In this chapter, we discuss the various symmetry properties that each of the factors obey. In particular, we observe a hierarchy amongst the various symmetries and emphasize the origins of each symmetry.

In Sections 2 and 3, the phase freedom and complex conjugation symmetry are given for the GH transformation factor. We show that the phase freedom factor and the complex conjugation factor, defined for the GH scheme, are archetypal, because they give corresponding factors for the coupling, subduction and induction schemes. In Section 4 we discuss the transposition symmetry which originates with the appearance of a direct product group $H \times K$ in the group-subgroup scheme, and in Section 5 we introduce the associative symmetry which arises from the inclusion of a triple product group $L \times M \times N$. The former defines the transposition factor and the latter leads to the associative factors. Evidence for the hierarachy of symmetries is shown by giving the phase freedom and complex conjugation symmetries of the transformation factors and associative factors. The associative factors have in addition symmetries due to transposition and cyclic permutations of the groups L,M,N and one further symmetry arising from a higher associative property of a direct product of four groups. We show the connection of the identities for the last two symmetries with the Racah backcoupling relation and the Biedenharn-Elliott sum rule, which are well-known identities in the Racah Wigner algebra.

Section 6 contains a similar discussion of the symmetries of the coupling, subduction and induction factors. We also give the analogous Wigner relations for the three schemes.

Finally, in sections 5 and 6 we discuss a recursive method for the calculation of the associative factors and the coupling, subduction and induction factors, based on the Butler method for the calculation of 6j and 3jm symbols in the Racah-Wigner algebra.

2. THE PHASE FREEDOM SYMMETRY

In this section we discuss what is meant by phase freedom in the group-subgroup (GH) transformation theory. Consider a GH basis, $\{|z\gamma a\eta j\rangle\}$. A transformation between inequivalent GH bases but equivalent H bases must have the form (see II.2.10-11)

$$|z\gamma a'\eta j\rangle = |z\gamma a\eta j\rangle\langle\gamma a\eta|\gamma a'\eta\rangle \quad . \qquad \text{(III.2.1)}$$

The factor $<\gamma a\eta|\gamma\alpha\ \eta>$ specifies the relationship between the inequivalent GH bases labelled by the branching multiplicity labels, a and a'. The phase freedom is such a transformation. It is a matrix describing the arbitrariness in choosing one GH basis $\{|z\gamma a\eta j>\}$ over an inequivalent GH basis $\{|z\gamma\hat{a}\eta j>\}$. We denote the phase freedom in a basis vector as

$$|z\gamma a\eta j> = |z\gamma\hat{a}\eta j>U(\gamma,\eta)^{\hat{a}}{}_{a} \quad . \qquad \text{(III.2.2)}$$

We shall call $U(\gamma,\eta)^{\hat{a}}{}_{a}$ the phase freedom factor. Although this equation is similar to (2.1), we point out that the phase freedom is a transformation in branching multiplicity labels only. To emphasise this, we write the phase freedom in the GHK basis $\{|z\gamma a\eta b\kappa k>\}$ as

$$|z\gamma a\eta b\kappa j> = |z\gamma\hat{a}\eta\hat{b}\kappa j>U(\gamma,\eta)^{\hat{a}}{}_{a}\ U(\eta,\kappa)^{\hat{b}}{}_{b} \,. \qquad \text{(III.2.3)}$$

(2.2) and (2.3) are the archetypes of all phase freedoms.

As a further illustration, the phase freedom occurring for the basis vectors, $\{|y\eta\uparrow a\gamma i\}$, of the induced representation space $V_{y\eta\uparrow}$ can be written as

$$|y\eta\uparrow a\gamma i> = |y\eta\uparrow\hat{a}\gamma i>\ U(\eta\uparrow,\gamma)^{\hat{a}}{}_{a} \qquad \text{(III.2.4)}$$

In analogy to (2.3), the basis vectors of the induction scheme ((L)↑H)↑G have the phase freedom

$$|x\lambda\uparrow a\eta\uparrow b\gamma i> = |x\lambda\uparrow\hat{a}\eta\uparrow\hat{b}\gamma i>\ U(\lambda\uparrow,\eta)^{\hat{a}}{}_{a}\ U(\eta\uparrow,\gamma)^{\hat{b}}{}_{b} \,. \qquad \text{(III.2.5)}$$

Both (2.4) and (2.5) are archetypal of phase freedoms for induction schemes. We shall introduce other phase freedom factors c ·responding to various coupling, subduction, and induction schemes in later sections.

3. THE COMPLEX CONJUGATION SYMMETRY

The theory of the complex conjugate irrep of $\gamma(G)$ is well documented (see Derome and Sharp 1965, Butler 1975). Bickerstaff (1980) has given a detailed account of the complex conjugation symmetry with particular application to any GH basis. The GH transformation factors were shown to obey an analogous relation to the Derome-Sharp lemma of the Racah-Wigner algebra. In this section we give a brief outline of this symmetry with the objective of giving archetypal relations for all GH transformation factors, including those of induction schemes. We follow the operator approach given by Bickerstaff (1980).

The complex conjugation operator k_γ is defined as an antilinear unitary operator, mapping an irrep space $V_{z\gamma}$ into the irrep space $V_{z^*\gamma^*}$, that is

$$k_\gamma : V_{z\gamma} \to V_{z^*\gamma^*}$$

$$k_\gamma \, |\mu>\alpha = |k_\gamma \, \mu> \alpha^* \qquad \text{for} \quad \alpha\varepsilon C \quad \mu\varepsilon V_{z\gamma} \qquad \text{(III.3.1)}$$

$$\text{and} \qquad k_\gamma^\dagger k_\gamma = e = k_\gamma k_\gamma^\dagger$$

where e is the identity operator. The action of k_γ on a G basis of $V_{z\gamma}$ defines the complex conjugate basis for $V_{z^*\gamma^*}$ which may not necessarily be a G basis. We have

$$\begin{aligned} k_\gamma \, |z\gamma i \; &\equiv \; |z^*\gamma^* i^*> \\ &= |z^*\gamma^* i'>A(\gamma)^{i'}{}_{i} \end{aligned} \qquad \text{(III.3.2)}$$

where the coefficient $A(\gamma)^{i'}{}_{i}$ describes the transformation from the basis $\{|z^*\gamma^* i^*>\}$ to the G basis of $V_{z^*\gamma^*}$. The coefficients are the 2jm symbols, usually written $\binom{\gamma^* \gamma}{i' i}$, of Derome and Sharp (1965) and Butler (1975, 1981), however we prefer to call them complex conjugation coefficients following Bickerstaff (1980). They have the unitary properties

$$A(\gamma)^{\dagger i}{}_{i'} \; A(\gamma)^{i'}{}_{j}{}^{*} \quad = \quad \delta^{i}{}_{j} \tag{III.3.3}$$

$$A(\gamma)^{i'}{}_{i} \quad A(\gamma)^{\dagger i}{}_{j'}{}^{*} \quad = \quad \delta^{i'}{}_{j'} \tag{III.3.4}$$

The complex conjugation sign on the right hand coefficient arises from the antilinearity of k_γ.

The definition of the complex conjugation operators applies also to the irrep space $V_{z'\gamma^*}$ with G basis $\{|z'\gamma^* i'\rangle\}$. The operator k_{γ^*} has the action

$$k_{\gamma^*} \; |z'\gamma^* i'\rangle \equiv |z'^*\gamma i'^*\rangle$$

$$= |z'^*\gamma i\rangle \quad A(\gamma^*)^{i}{}_{i'} \tag{III.3.5}$$

The product $k_{\gamma^*} k_\gamma$ is a linear operator which by the matrix invariance requirement has the property

$$k_{\gamma^*} k_\gamma = \{\gamma\} e \tag{III.3.6}$$

where e is the identity operator and $\{\gamma\}$ the 1j symbol satisfying $\{\gamma^*\} = \{\gamma\}^*$. The symmetry properties of the complex conjugation coefficients follow from the definitions of k_γ and k_{γ^*} and (3.6)

$$A(\gamma^*)^{i}{}_{i'} \quad = \quad \{\gamma\} A(\gamma)^{\dagger i}{}_{i'} \tag{III.3.7}$$

$$A(\gamma)^{\dagger i}{}_{i'} \quad = \quad A(\gamma)^{i'}{}_{i} \tag{III.3.8}$$

We have assumed here that the relationship between the parentage labels z and z* is the same as that between

z' and z'^*.

Using k_γ we can express any transformation between alternate G bases of $V_{z^*\gamma^*}$ as a transformation between G bases of $V_{z\gamma}$,

$$\langle\gamma^*i'|\gamma^*j'\rangle = A(\gamma)^{i'}{}_{i} \langle\gamma i|\gamma j\rangle^* A(\gamma)^{\dagger j}{}_{j'}{}^* \qquad \text{(III.3.9)}$$

where the antilinearity of k_γ gives rise to the complex conjugate signs on the right hand side. This is the complex conjugation symmetry of any transformation coefficient since it relates the coefficients of the transformation in the irrep space $V_{z'\gamma^*}$ to the complex conjugated coefficients of the transformation in $V_{z\gamma}$. We can apply this relation not only to a single group G but to direct product groups. In particular, it is applicable to the coupling coefficient for which the symmetry is known as the Derome-Sharp lemma (cf. Stedman 1976, Bickerstaff 1980).

Consider a GH basis. We define the complex conjugation operator $k_{\gamma\eta}$ as

$$\begin{aligned} k_{\gamma\eta} |z\gamma a\eta j\rangle &\equiv |z^*\gamma^*a^*\eta^*j^*\rangle \\ &= |z^*\gamma^*a^*\eta^*j'\rangle \; A(\eta)^{j'}{}_{j} \\ &= |z^*\gamma^*a'\eta^*j'\rangle \; A(\gamma,\eta)^{a'}{}_{a} \; A(\eta)^{j'}{}_{j} \end{aligned} \qquad \text{(III.3.10)}$$

where we have used (3.5) to transform from the complex conjugate basis to a H basis giving the complex conjugation coefficient of H, and then equations (II.2.10-11) in transforming to an inequivalent GH basis but equivalent H basis of $V_{z^*\gamma^*}$. This last transformation defines the complex conjugation factor $A(\gamma,\eta)^{a'}{}_{a}$ which has the following properties

$$A(\gamma,\eta)^{\dagger a}{}_{a'} \quad A(\gamma,\eta)^{a'}{}_{b}{}^{*} = \delta^{a}{}_{b} \tag{III.3.11}$$

$$A(\gamma,\eta)^{a'}{}_{a} \quad A(\gamma,\eta)^{\dagger a}{}_{b'}{}^{*} = \delta^{a'}{}_{b'} \tag{III.3.12}$$

$$A(\gamma,\eta)^{\dagger a}{}_{a'} = A(\gamma,\eta)^{a'}{}_{a} \tag{III.3.13}$$

and $$A(\gamma^*,\eta^*)^{a}{}_{a'} = \{\gamma\}\{\eta\}^* \, A(\gamma,\eta)^{\dagger a}{}_{a'} \tag{III.3.14}$$

In addition, the complex conjugation factor can be used to obtain the complex conjugation symmetry for any transformation factor. We have

$$\langle\gamma^*b'\eta^*|\gamma^*a'\eta^*\rangle = A(\gamma,\eta)^{b'}{}_{b}\langle\gamma b\eta|\gamma a\eta\rangle^{*}A(\gamma,\eta)^{\dagger a}{}_{a'}{}^{*} \tag{III.3.15}$$

which is the archetype for all GH transformation factors.

Bickerstaff (1980) has discussed the possible choices of the complex conjugation factor. With the phase freedom, in matrix form,

$$\hat{A}(\gamma,\eta) = U(\gamma^*,\eta^*)A(\gamma,\eta)U(\gamma,\eta)^{T} \tag{III.3.16}$$

and the symmetry properties (3.11-14), one possible set of choices is

(1) if γ or η is a complex irrep, $A(\gamma,\eta) = I$ (III.3.17)

(2) if γ and η are both real then

(a) if $\{\gamma\}\{\eta\} = +1$ $\quad A(\gamma,\eta) = I$ (III.3.18)

(b) if $\{\gamma\}\{\eta\} = -1$ $\quad A(\gamma,\eta) = J$ (III.3.19)

where $J \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes I$. In requiring that any other change of basis in the branching label does not change the above choices of the complex conjugate factors, we must constrain the phase freedom by

$$U(\gamma^*,\eta^*) = A(\gamma,\eta)\ U(\gamma,\eta)^*\ A(\gamma,\eta)^{T} \ . \tag{III.3.20}$$

That is, for (2a) above, the phase freedom must be real and hence orthogonal, while for (2b), it must satisfy $U(\gamma,\eta)^* = - J\, U(\gamma,\eta)\, J$.

We now consider the induction scheme. Using (3.7), we define the complex conjugation operator, $k_{\eta\uparrow,\gamma}$, acting on the basis vectors of the induced space $V_{\eta\uparrow a\gamma}$ as

$$k_{\eta\uparrow\gamma}\ |y\eta\uparrow a\gamma i\rangle \equiv |y^*\eta^*\uparrow a^*\gamma^* i^*\rangle$$

$$= |y^*\eta^*\uparrow a'\gamma^* i'\rangle\ A(\eta\uparrow,\gamma)^{a'}{}_{a}\ A(\gamma)^{i'}{}_{i} \qquad \text{(III.3.21)}$$

where $A(\eta\uparrow,\gamma)^{a'}{}_{a}$ is the complex conjugation factor transforming between the complex conjugate basis and the GH basis.

$$A(\eta\uparrow,\gamma)^{\dagger a}{}_{a'}\ A(\eta\uparrow,\gamma)^{a'*}{}_{b} = \delta^{a}{}_{b} \qquad \text{(III.3.22)}$$

$$A(\eta\uparrow,\gamma)^{a'}{}_{a}\ A(\eta\uparrow,\gamma)^{\dagger a}{}_{b'}{}^{*} = \delta^{a'}{}_{b'} \qquad \text{(III.3.23)}$$

$$A(\eta\uparrow,\gamma)^{\dagger a}{}_{a'} = A(\eta\uparrow,\gamma)^{a'}{}_{a} \qquad \text{(III.3.24)}$$

and $$A(\eta^*\uparrow,\gamma^*)^{a}{}_{a'} = \{\eta\}\{\gamma\}^* A(\eta\uparrow,\gamma)^{a'}{}_{a}\ . \qquad \text{(III.3.25)}$$

We also obtain the complex conjugation symmetry of transformation factors, $\langle\eta\uparrow b\gamma|\eta\uparrow a\gamma\rangle$,

$$\langle\eta^*\uparrow b'\gamma^*|\eta^*\uparrow a'\gamma^*\rangle = A(\eta\uparrow,\gamma)^{b'}{}_{b}\ \langle\eta\uparrow b\gamma|\eta\uparrow a\gamma\rangle^*\ A(\eta\uparrow,\gamma)^{\dagger a*}{}_{a'}\ .$$

(III.3.26)

Direct product groups may be substituted into the schemes $G\supset H$ and $H\uparrow G\supset G$, to give other complex conjugation factors. We list those that will appear in later sections along with the symmetry corresponding to $k_{\gamma^*} = \{\gamma\}k_{\gamma}^{\dagger}$.

(1) $\eta\kappa(H\times K)\supset\gamma(G)$: $A(\eta\kappa,\gamma)^{a'}{}_{a}$

$$A(\eta^*\kappa^*,\gamma^*)^{a}{}_{a'} = \{\eta\}\{\kappa\}\{\gamma\}^*A(\eta\kappa,\gamma)^{a'}{}_{a} \qquad (III.3.27)$$

(2) $\gamma(G)\supset\eta\kappa(H\times K)$: $A(\gamma,\eta\kappa)^{a'}{}_{a}$

$$A(\gamma^*,\eta^*\kappa^*)^{a}{}_{a'} = \{\gamma\}\{\eta\}^*\{\kappa\}^*A(\gamma,\eta\kappa)^{a'}{}_{a} \qquad (III.3.28)$$

(3) $\eta\kappa(H\times K)\uparrow G\supset\gamma(G)$: $A(\eta\kappa\uparrow,\gamma)^{a'}{}_{a}$

$$A(\eta^*\kappa^*\uparrow,\gamma^*)^{a}{}_{a'} = \{\eta\}\{\kappa\}\{\gamma\}^*A(\eta\kappa\uparrow,\gamma)^{a'}{}_{a} \qquad (III.3.29)$$

These complex conjugation factors can be chosen to have the same matrix structure as in (3.17-19). However, to coincide with Butler's choices of permutation matrices (Butler 1975, see §4), we treat the product irrep $\eta\eta^*(H\times H)$, for η complex, as a real irrep. Hence, the 1j phase $\{\gamma\}$ determines whether the complex conjugation factor is I or J when it could equally have been chosen unity.

As a consequence of maintaining these choices of complex conjugation factors, the phase freedom appropriate to these schemes is restricted in accordance with (3.20). Hence

$$\begin{aligned}
&(i) & U(\eta^*\kappa^*,\gamma^*) &= A(\eta\kappa,\gamma)U(\eta\kappa,\gamma)^*A(\eta\kappa,\gamma)^T \\
&(ii) & U(\gamma^*,\eta^*\kappa^*) &= A(\gamma,\eta\kappa)U(\gamma,\eta\kappa)^*A(\gamma,\eta\kappa)^T \qquad (III.3.30) \\
&(iii) & U(\eta^*\kappa^*\uparrow,\gamma^*) &= A(\eta\kappa\uparrow,\gamma)U(\eta\kappa\uparrow,\gamma)^*A(\eta\kappa\uparrow,\gamma)^T .
\end{aligned}$$

4. THE TRANSPOSITION SYMMETRY

The appearance of a direct product group $H\times K$ say, in a group-subgroup decomposition leads to a further symmetry, that of transposing the two groups H and K. When the two groups are the same the symmetry is non-trivial. We include a definition of the transposition with respect to induced representations.

The three group-subgroup schemes, which reflect the transposition symmetry, are those corresponding to bases

$$\text{(i)} \quad \{|xy\ \eta\kappa(H\times K)\ z\ \gamma(G)\ i>\}$$

$$\text{(ii)} \quad \{|z\ \gamma(G)\ a\ \eta\kappa(H\times K)\ jk>\} \qquad \text{(III.4.1)}$$

and

$$\text{(iii)} \quad \{|xy\ \eta\kappa(H\times K)\uparrow G\ a\gamma(G)\ i>\}$$

The transposition operator τ is defined as an involutary, unitary, linear operator, which acts on the basis vectors to transpose the irrep labels η and κ. We have

$$\text{(i)} \quad \tau\ |xy\eta\kappa a\gamma i> = |yx\kappa\eta a'\gamma i><\kappa\eta a'\gamma|\tau|\eta\kappa a\gamma>$$

$$\text{(ii)} \quad \tau\ |z\gamma a\eta\kappa jk> = |z\gamma a'\kappa\eta kj><\gamma a'\kappa\eta|\tau|\gamma a\eta\kappa> \quad \text{(III.4.2)}$$

$$\text{(iii)} \quad \tau\ |xy\eta\kappa\uparrow a\gamma i> = |yx\kappa\eta\uparrow a'\gamma i><\kappa\eta\uparrow a'\gamma|\tau|\eta\kappa\uparrow a\gamma> .$$

The transposition operator is also required to transform between inequivalent GH bases and equivalent H bases for the subgroup. The resulting factorization defines the three transposition factors, which we will use in this and later sections. For notational compactness, we shall denote them in the matrix element form

$$\text{(i)}\quad T(\eta\kappa,\gamma)^{a'}{}_{a}\ ,\quad \text{(ii)}\quad T(\gamma,\eta\kappa)^{a'}{}_{a}\ ,\quad \text{(iii)}\quad T(\eta\kappa\dagger,\gamma)^{a'}{}_{a}$$

(III.4.3)

There are two symmetries of the transposition factors. The first of these is a consequence of the involutary nature of τ. This gives the following relations,

$$\text{(1) (i)}\quad T(\kappa\eta,\gamma) = T(\eta\kappa,\gamma)^{\dagger}$$

$$\text{(ii)}\quad T(\gamma,\kappa\eta) = T(\gamma,\eta\kappa)^{\dagger} \qquad \text{(III.4.4)}$$

$$\text{(iii)}\quad T(\kappa\eta\dagger,\gamma) = T(\eta\kappa\dagger,\gamma)^{\dagger}$$

Note that for the case H=K and $\eta=\kappa$, the transposition factor is hermitian. The second symmetry is the complex conjugation symmetry, which must be satisfied by all transformation factors. From (3.15) and (3.26), with appropriate substitution of the group and subgroup labels, we have

$$\text{(2) (i)}\quad T(\eta^*\kappa^*,\gamma^*) = A(\kappa\eta,\gamma)T(\eta\kappa,\gamma)^*A(\eta\kappa,\gamma)^{\dagger *}$$

$$\text{(ii)}\quad T(\gamma^*,\eta^*\kappa^*) = A(\gamma,\kappa\eta)T(\gamma,\eta\kappa)^*A(\gamma,\eta\kappa)^{\dagger *} \qquad \text{(III.4.5)}$$

$$\text{(iii)}\quad T(\eta^*\kappa^*\dagger,\gamma^*) = A(\kappa\eta\dagger,\gamma)T(\eta\kappa\dagger,\gamma)^*A(\eta\kappa\dagger,\gamma)^{\dagger *}\ .$$

Special attention must be given to two cases:

(1) if K=H, $\kappa=\eta^*\neq\eta$ and $\gamma=\gamma^*$, we can combine (4.4-5) with (3.27-29) to give

$$\text{(i)}\quad T(\eta^*\eta,\gamma)A(\eta\eta^*,\gamma)=\{\gamma\}[T(\eta^*\eta,\gamma)A(\eta\eta^*,\gamma)]^T$$

$$\text{(ii)}\quad T(\gamma,\eta^*\eta)A(\gamma,\eta\eta^*)=\{\gamma\}[T(\gamma,\eta^*\eta)A(\gamma,\eta\eta^*)]^T \qquad \text{(III.4.6)}$$

$$\text{(iii)}\quad T(\eta^*\eta\dagger,\gamma)A(\eta\eta^*\dagger,\gamma) = \{\gamma\}[T(\eta^*\eta\dagger,\gamma)A(\eta\eta^*\dagger,\gamma)]^T \quad .$$

That is, the product, TA, is symmetric if γ is orthogonal or skew-symmetric if γ is symplectic. If the choices (3.17-19) of the complex conjugation factors are used, then for the former case, T is symmetric while, for the latter case, T satisfies $TJ = -(TJ)^T$.

(2) if all the irreps are real, the transposition factor satisfies $T = AT^*A^T$ which, with the choices of (3.17-19), implies that (i) if γ is orthogonal, T is real, and (ii) if γ is symplectic, the transposition factor satisfies $T = JT^*J^T$.

For all other cases, the symmetries given by (4.4-5) relate distinct transposition factors.

To obtain the choices for the transposition factors we must establish their phase freedom. We have

$$\text{(i)}\quad \hat{T}(\eta\kappa,\gamma) = U(\kappa\eta,\gamma)T(\eta\kappa,\gamma)U(\eta\kappa,\gamma)^\dagger$$

$$\text{(ii)}\quad \hat{T}(\gamma,\eta\kappa) = U(\gamma,\kappa\eta)T(\gamma,\eta\kappa)U(\gamma,\eta\kappa)^\dagger \qquad \text{(III.4.7)}$$

$$\text{(iii)}\quad \hat{T}(\eta\kappa\dagger,\gamma) = U(\kappa\eta\dagger,\gamma)T(\eta\kappa\dagger,\gamma)U(\eta\kappa\dagger,\gamma)^\dagger$$

If the complex conjugation choices are to remain invariant, the phase freedoms must be constrained by (3.30). However, there is still sufficient freedom to choose all transposition factors diagonal:

(1) for $\eta \neq \kappa$ (i) $\hat{T}(\eta\kappa,\gamma) = (\eta\kappa,\gamma)\ \mathbb{I}$

$$\text{(ii)}\quad \hat{T}(\gamma,\eta\kappa) = (\gamma,\eta\kappa)\ \mathbb{I} \qquad \text{(III.4.8)}$$

$$\text{(iii)}\quad \hat{T}(\eta\kappa\dagger,\gamma) = (\eta\kappa\dagger,\gamma)\ \mathbb{I}$$

where $(\eta\kappa,\gamma)$, $(\gamma,\eta\kappa)$ and $(\eta\kappa\dagger,\gamma)$ are phases which have been chosen to have no dependency on the multiplicity labels.

(2) for $\eta=\kappa$ (i) $\hat{T}(\eta\eta,\gamma) = (\eta\eta,\kappa)\,\mathbb{H}$

$$\text{(ii)}\quad \hat{T}(\gamma,\eta\eta) = (\gamma,\eta\eta)\,\mathbb{H} \qquad \text{(III.4.9)}$$

$$\text{(iii)}\quad \hat{T}(\eta\eta,\dagger\gamma) = (\eta\eta\dagger,\gamma)\,\mathbb{H}$$

where $(\eta\eta,\gamma)$, $(\gamma,\eta\eta)$ and $(\eta\eta\dagger,\gamma)$ equal ± 1, and

$$\mathbb{H} = \begin{pmatrix} 1_s & 0 \\ 0 & -1_a \end{pmatrix}$$ (s and a are obtained from character theory, see Derome (1965)).

For the choices (4.8) and (4.9) to remain invariant under further choices of symmetry, the following phase restrictions must be imposed

$$\text{(i)}\quad U(\kappa\eta,\gamma) = T(\eta\kappa,\gamma)U(\eta\kappa,\gamma)T(\eta\kappa,\gamma)^{\dagger}$$

$$\text{(ii)}\quad U(\gamma,\kappa\eta) = T(\gamma,\eta\kappa)U(\gamma,\eta\kappa)T(\gamma,\eta\kappa)^{\dagger} \qquad \text{(III.4.10)}$$

$$\text{(iii)}\quad U(\kappa\eta\dagger,\gamma) = T(\eta\kappa\dagger,\gamma)U(\eta\kappa\dagger,\gamma)T(\eta\kappa\dagger,\gamma)^{\dagger}\ .$$

For $\eta=\kappa$ and the above choice of transposition factor, the phase freedom is restricted to the block diagonal form

$$U = \begin{pmatrix} U_s & o \\ o & U_a \end{pmatrix}\ . \qquad \text{(III.4.11)}$$

5. THE ASSOCIATIVE SYMMETRY

A further symmetry arises from the direct product of three groups, L, M and N, for which there are two ways of associating pairs of groups given the ordering $L\times M\times N$, namely $((L\times M)\times N)$ and $(L\times(M\times N))$. The process by which we associate these groups, can be either: (i) coupling, (ii) subduction, or (iii) induction. The associative symmetry gives two different group-subgroup schemes for labelling the direct product irrep of $\lambda(L)$, $\mu(M)$ and $\nu(N)$ (see figures II.2.3, II.2.4, II.5.3). The overlap between the two basis vectors defines the recoupling, resubduction and reinduction

$$\text{(i)}\quad \langle(\lambda\mu)a\eta,\nu,b\gamma|\lambda(\mu\nu)c\kappa,d\gamma\rangle$$

$$\text{(ii)}\quad \langle\gamma a\eta(b\eta\mu)\nu|\gamma c\lambda\kappa(d\mu\nu)\rangle \tag{III.5.1}$$

$$\text{(iii)}\quad \langle(\lambda\mu)\uparrow a\eta,\nu,\uparrow b\gamma|\lambda(\mu\nu)\uparrow c\kappa,\uparrow d\gamma\rangle$$

These three factors, which we shall collectively call the associative factors, have similar symmetry properties.
The complex conjugation symmetry is obtained from (3.15) and (3.26).

$$\text{(i)}\quad \langle(\lambda^*\mu^*)a'\eta^*,\nu^*,b'\gamma^*|\lambda^*(\mu^*\nu^*)c'\kappa^*,d'\gamma^*\rangle$$

$$= A(\lambda\mu,\eta)^{a'}{}_{a}\; A(\eta\nu,\gamma)^{b'}{}_{b}$$

$$\langle(\lambda\mu)a\eta,\nu,b\gamma|\lambda(\mu\nu)c\kappa,d\gamma\rangle^*$$

$$A(\mu\nu,\kappa)^{\dagger c}{}_{c'}{}^{*}\; A(\eta\kappa,\gamma)^{\dagger d}{}_{d'}{}^{*}$$

$$\text{(ii)}\quad \langle\gamma^*a'\eta^*(b'\lambda^*\mu^*)\nu^*|\gamma^*c'\lambda^*\kappa^*(d'\mu^*\nu^*)\rangle$$

$$= A(\gamma,\eta\nu)^{a'}{}_{a}\; A(\eta,\lambda\mu)^{b'}{}_{b}$$

$$\langle\gamma a\eta(b\lambda\mu)\nu|\gamma c\lambda\kappa(d\mu\nu)\rangle^* \tag{III.5.2}$$

$$A(\gamma,\lambda\kappa)^{\dagger c}{}_{c'}{}^{*}\; A(\kappa,\mu\nu)^{\dagger d}{}_{d'}{}^{*}$$

(iii) $<(\lambda^*\mu^*)\dagger a'r^*,\nu^*,\dagger b'\gamma^*|\lambda^*(\mu^*\nu^*(\dagger c'\kappa^*,\dagger d'\gamma^*>$

$$= A(\lambda\mu\dagger,\eta)^{a'}{}_{a} \quad A(\eta\nu\dagger,\gamma)^{b'}{}_{b}$$

$$<(\lambda\mu)\dagger a\eta,\nu,\dagger b\gamma|\lambda(\mu\nu)\dagger c\kappa,\dagger d\gamma>^*$$

$$A(\mu\nu\dagger,\kappa)^{\dagger c}{}_{c'}{}^{*} \quad A(\eta\kappa\dagger,\gamma)^{\dagger d}{}_{d'}{}^{*}$$

Transformations are performed between inequivalent GH bases for both subgroup and intermediate groups (see 3.10-3.21). The result is the diagonalisation of the complex conjugation factor in the irrep labels of the intermediate group. For example, the first pair of complex conjugation factors from each of the above equations is given by

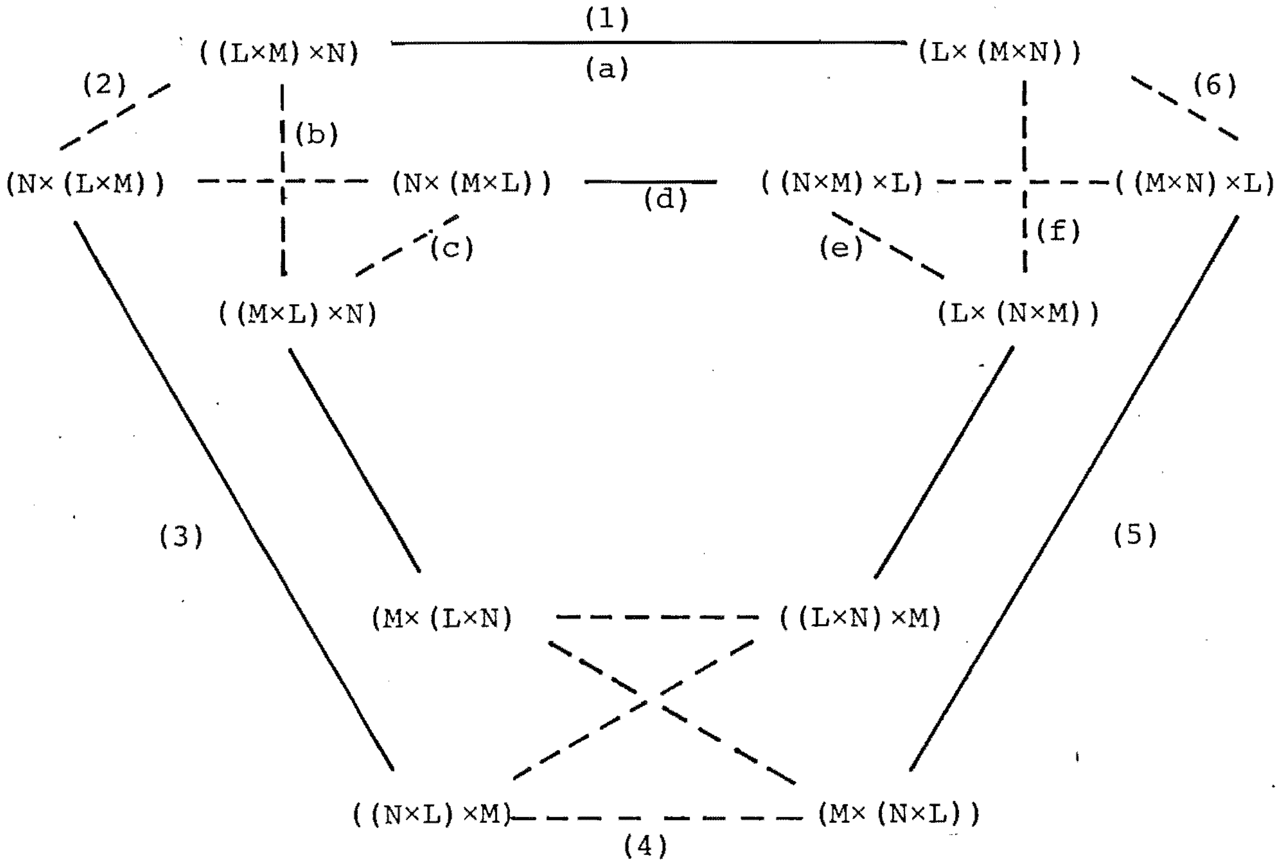
$$\text{(i)} \quad A(\lambda\mu\nu,\gamma)^{a'\eta'b'}_{a\eta b} = A(\lambda\mu,\eta)^{a'}{}_{a} \, \delta^{\eta'}{}_{\eta^*} \, A(\eta\nu,\gamma)^{b'}{}_{b}$$

$$\text{(ii)} \quad A(\gamma,\lambda\mu\nu)^{a'\eta'b'}_{a\eta b} = A(\gamma,\eta\nu)^{a'}{}_{a} \, \delta^{\eta'}{}_{\eta^*} \, A(\eta,\lambda\mu)^{b'}{}_{b} \qquad \text{(III.5.3)}$$

$$\text{(iii)} \quad A(\lambda\mu\nu\dagger,\gamma)^{a'\eta'b'}_{a\eta b} = A(\lambda\mu\dagger,\eta)^{a'}{}_{a} \, \delta^{\eta'}{}_{\eta^*} \, A(\eta\nu\dagger,\gamma)^{b'}{}_{b}$$

The transposition symmetry may be used to develop two further identities. Their derivation can be seen more clearly by the schematic diagram (figure 5.1), which gives the relationship between the 2×3! possible ways of associating L, M and N under permutation. The solid lines denote associative factors and the broken lines denote transposition factors. In traversing the paths labelled (a) to (f) and (1) to (6), we obtain two identities which give certain permutation symmetries of the associative factors:

Figure III.5.1



(1) the λ-ν transposition symmetry,

(i) $$\langle\nu(\mu\lambda)a'\eta,b'\gamma|(\mu\nu)c'\kappa,\lambda,d'\gamma\rangle = T(\eta\mu,\eta)^{a'}{}_{a}\ T(\eta\nu,\gamma)^{b'}{}_{b}$$

$$\times\langle(\lambda\mu)a\eta,\nu,b\gamma|\lambda(\mu\nu)c\kappa,d\gamma\rangle\ T(\mu\nu,\kappa)^{\dagger c}{}_{c'}\ T(\lambda\kappa,\gamma)^{\dagger d'}{}_{d}$$

(ii) $$\langle\gamma a'\nu\eta(b'\mu\lambda)|\gamma c'\kappa(d'\nu\mu)\lambda\rangle = T(\gamma,\eta\nu)^{a'}{}_{a}\ T(\eta,\lambda\mu)^{b'}{}_{b}$$

$$\times\langle\gamma a\eta(b\lambda\mu)\nu|\gamma c\lambda\kappa(d\mu\nu)\rangle\quad T(\gamma,\lambda\kappa)^{\dagger c}{}_{c'}\ T(\kappa,\mu\nu)^{\dagger d}{}_{d'}$$

(iii) $$\langle\nu(\mu\lambda)\uparrow a'\eta,\uparrow b'\gamma|(\nu\mu)\uparrow c'\kappa,\lambda,\uparrow d'\gamma\rangle = T(\lambda\mu\uparrow,\eta)^{a'}{}_{a}\,T(\eta\nu\uparrow,\gamma)^{b'}{}_{b}$$

$$\times\langle(\lambda\mu)\uparrow a\eta,\nu,\uparrow b\gamma\uparrow\lambda(\mu\nu)\uparrow c\kappa,\uparrow d\gamma\rangle\ T(\mu\nu\uparrow,\kappa)^{\dagger c}{}_{c'}\ T(\lambda\kappa\uparrow,\gamma)^{\dagger d'}{}_{d}$$

and (2), the λ-μ- cylic symmetry,

$$\text{(i)}\quad T(\eta\nu,\gamma)^{b'}{}_{b_2}\ \langle(\lambda\mu)a'\eta,\nu,b_2\gamma|\lambda(\mu\nu)c_1\kappa,d_1\gamma\rangle$$

$$\times T(\kappa\lambda,\gamma)^{d_1}{}_{d_2}\ \langle(\mu\nu)c_1\kappa,\lambda,d_2\gamma|\mu(\nu\lambda)b_1\rho,a\gamma\rangle$$

$$\times T(\rho\mu,\gamma)^{a_1}{}_{a_2}\langle(\nu\lambda)b_1\rho,\mu,a_2\gamma|\nu(\lambda\mu)a\eta,b\gamma\rangle$$

$$= \delta^{a'}{}_{a}\ \delta^{b'}{}_{b}$$

$$\text{(ii)}\quad T(\gamma,\eta\nu)^{a'}{}_{a_2}\langle\gamma a_2\eta(b'\eta_\mu)\nu|\gamma c_1\lambda\kappa(d_1\mu\nu)\rangle$$

$$\times T(\gamma,\kappa\lambda)^{c_1}{}_{c_2}\langle\gamma c_2\kappa(d_1\mu\nu)\lambda|\gamma b_1\mu\rho(a_1\nu\lambda)\rangle \qquad \text{(III.5.5)}$$

$$\times T(\gamma,\rho\mu)^{b_1}{}_{b_2}\ \langle\gamma b_2\rho(a_1\nu\lambda)\mu|\gamma a\nu\eta(b\lambda\mu)\rangle$$

$$= \delta^{a'}{}_{a}\ \ \delta^{b'}{}_{b}$$

$$\text{(iii)}\quad T(\eta\nu\uparrow,\gamma)^{b'}{}_{b_2}\langle(\lambda\mu)\uparrow a'\eta,\nu,\uparrow b_2\gamma|\lambda(\mu\nu)\uparrow c_1\kappa,\uparrow d_1\gamma\rangle$$

$$\times T(\eta\nu\uparrow,\gamma)^{d_1}{}_{d_2}\ \langle(\lambda\mu)\uparrow c_1\kappa,\lambda,\uparrow d_2\gamma|\mu(\nu\lambda)\uparrow b_1\rho,\uparrow a_1\gamma\rangle$$

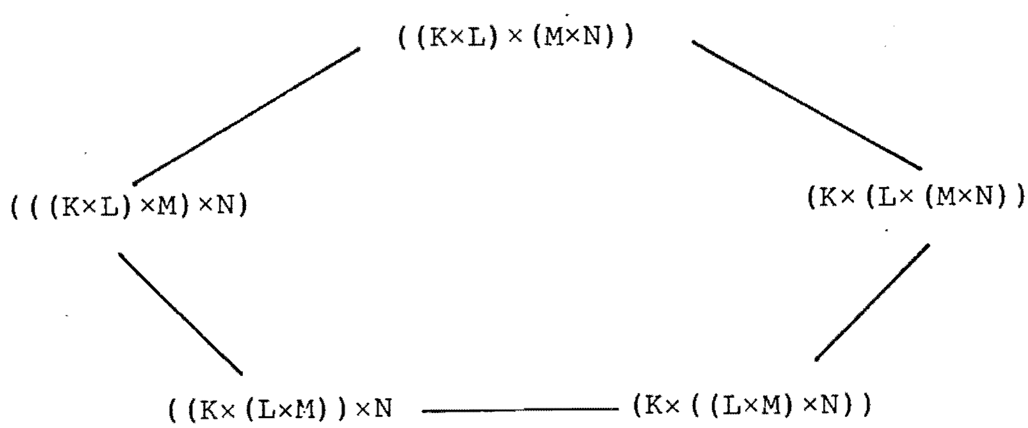
$$\times T(\rho\mu\uparrow,\gamma)^{a_1}{}_{a_2}\langle(\nu\lambda)\uparrow b_1\rho,\mu,\uparrow a_2\gamma|\nu(\lambda\mu)\uparrow a\eta,\uparrow b\gamma\rangle$$

$$= \delta^{a'}{}_{a}\ \delta^{b'}{}_{b}\quad .$$

The transposition factors are factorised in a similar manner to the complex conjugation factors of (5.3).

A similar factorization consideration occurs for particular transformation factors of direct product groups of more than three groups, and leads to one further important identity. A direct product of N groups can be formed by associating pairs of groups. The number of such associative N-product schemes, not including permutations of the groups, is given by the recursive formula $a(N) = \sum_{I=1}^{N-1} a(I)a(N-I)$ with $a(0)\equiv a(1)\equiv 1$. For n=3,4,5, we have a (N)=2,5,14 respectively. However a transformation factor between basis vectors of two different associative N-product scheme can be written as a product of transformation factors of an associative 3-product scheme, that is a product of associative factors. We give a specific example. For the direct product of four groups, there are five associative schemes (see figure 5.2). These schemes are arranged in a particular order. Adjacent schemes, connected by a solid line, involve a common associative pair, for example $(K\times L)$ is the common associative pair of $(((K\times L)\times M)\times N)$ and $((K\times L)\times(M\times N))$. Using the matrix invariance requirement, we can factor the common associative pair from the transformation factor relating these two schemes. In this way, a transformation factor between $(((K\times L)\times M)\times N)$ and $(K\times(L\times(M\times N)))$, for example, can be written as a product of two associative factors with $((K\times L)\times(M\times N))$ as the intermediate associative scheme. However, there is an alternative route involving the two lower associative schemes of figure 5.2. This leads to an important identity for the associative factors:

<u>Figure III.5.2</u>



$$
\begin{aligned}
\text{(i)}\quad & \langle(\rho\mu)a\eta,\nu,b\gamma|\rho(\mu\nu)c\tau,d\gamma\rangle \\
& \times\langle(\kappa\lambda)c'\rho,\tau,d\gamma|\kappa(\lambda\tau)a'\varepsilon,b'\gamma\rangle \\
& =\langle(\kappa\lambda)c'\rho,\mu,a\eta|\kappa(\lambda\mu)b_1\sigma,d_1\eta\rangle \\
& \times\langle(\kappa\sigma)d_1\eta,\nu,b\gamma|\kappa(\sigma\nu)d_2\varepsilon,b'\gamma\rangle \\
& \times\langle(\lambda\mu)b_1\sigma,\nu,d_2\varepsilon|\lambda(\mu\nu)c\tau,a'\varepsilon\rangle
\end{aligned}
$$

$$
\begin{aligned}
\text{(ii)}\quad & \langle\gamma a\eta(b\rho\mu)\nu|\gamma c\rho\tau(d\mu\nu)\rangle \\
& \times\langle\gamma c\rho(d'\kappa\lambda)\tau|\gamma a'\kappa\varepsilon(b'\lambda\tau)\rangle \\
& =\langle\eta b\rho(d'\kappa\lambda)\mu|\eta c_1\kappa\sigma(a_1\lambda\mu)\rangle \\
& \times\langle\gamma a\eta(c_1\kappa\sigma)\nu|\gamma a'\kappa\varepsilon(c_2\sigma\nu)\rangle \\
& \times\langle\varepsilon c_2\sigma(a_1\lambda\mu)\nu|\varepsilon b'\lambda\tau(d\mu\nu)\rangle
\end{aligned}
\qquad \text{(III.5.6)}
$$

$$
\begin{aligned}
\text{(iii)}\quad & \langle(\rho\mu)\uparrow a\eta,\nu,\uparrow b\gamma|\rho(\mu\nu)\uparrow c\tau,\uparrow d\gamma\rangle \\
& \times\langle(\kappa\lambda)\uparrow c'\rho,\tau,\uparrow d\gamma|\kappa(\lambda\tau)\uparrow a'\varepsilon,\uparrow b'\gamma\rangle \\
& =\langle(\kappa\lambda)\uparrow c'\rho,\mu,\uparrow a\eta|\kappa(\lambda\mu)\uparrow b_1\sigma,\uparrow d_1\eta\rangle \\
& \times\langle(\kappa\sigma)\uparrow d_1\eta,\nu,\uparrow b\gamma|\kappa(\sigma\nu)\uparrow d_2\varepsilon,\uparrow b'\gamma\rangle \\
& \times\langle(\lambda\mu)\uparrow d_2\sigma,\nu,\uparrow\varepsilon|\lambda(\mu\nu)\uparrow c\tau,\uparrow a'\varepsilon\rangle \quad .
\end{aligned}
$$

The importance of these three identities, and those giving the symmetries of the associative factors, lies in their use in a recursive method for calculating the values of these factors. In particular, if we consider the coupling scheme with all groups isomorphic to each other, that is $L\simeq M\simeq N\simeq\ldots\simeq G$, then equations (5.2i), (5.4i), (5.5i) and (5.6i) are all well-known in the Racah-Wigner algebra. The recoupling factor is proportional to the higher symmetry 6j symbol (see Butler 1975, eqn. 9.13). The corresponding 6j versions of (5.2i), (5.4i), (5.5i) and (5.6i) are, respectively, the complex conjugation symmetry of the 6j, the row-flip symmetry of the 6j, the Racah back-coupling relation, and the Biedenharn-Elliott sum rule. These 6j relations have been used extensively in the building-up method developed by Butler (see Butler and Wybourne 1976, Butler 1981) for the calculation of 6j symbols, both for finite groups (Butler and Reid 1979, Butler 1981) and compact continuous groups (Butler, Haase and Wybourne 1978, Bickerstaff, Butler, Butts, Haase and Reid 1982, see also Chapter IV).

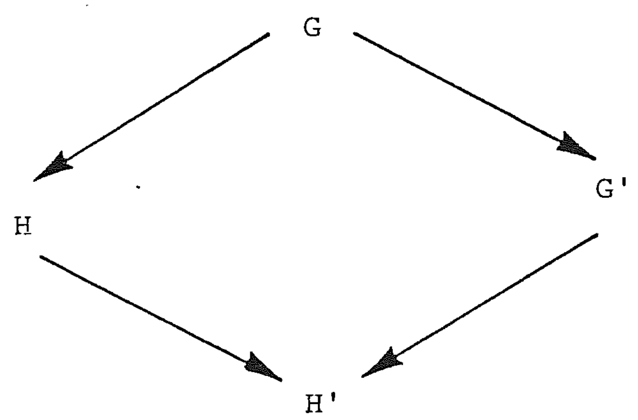
The building-up method could be applied to all associative factors. In such a calculation, the phase freedom of the factor must be determined, and this can be obtained using (2.3) and (2.5). We have

$$\begin{aligned}
&\text{(i)} \quad \langle(\lambda\mu)\hat{a}\eta,\nu,\hat{b}\gamma|\lambda(\mu\nu)\hat{c}\kappa,d\gamma\rangle = U(\lambda\mu,\eta)^{\hat{a}}{}_{a}\, U(\eta\nu,\gamma)^{\hat{b}}{}_{b} \\
&\qquad \times\langle(\lambda\mu)a\eta,\nu,b\gamma|\lambda(\mu\nu)c\kappa,d\gamma\rangle\, U(\mu\nu,\kappa)^{\dagger c}{}_{\hat{c}}\, U(\eta\kappa,\gamma)^{\dagger d}{}_{\hat{d}} \\
&\text{(ii)} \quad \langle\gamma\hat{a}\eta(\hat{b}\lambda\mu)\nu|\gamma\hat{c}\lambda\kappa(\hat{d}\mu\nu)\rangle = U(\gamma,\eta\nu)^{\hat{a}}{}_{a}\, U(\eta,\lambda\mu)^{\hat{b}}{}_{b} \qquad \text{(III.5.7)} \\
&\qquad \times\langle a\eta(b\lambda\mu)\nu|\gamma c\lambda\kappa(d\mu\nu)\rangle\, U(\gamma,\lambda\kappa)^{\dagger c}{}_{\hat{c}}\, U(\kappa,\mu\nu)^{d}{}_{\hat{d}} \\
&\text{(iii)} \quad \langle(\lambda\mu)\uparrow a\eta,\nu,\uparrow\hat{b}\gamma|\lambda(\mu\nu)\uparrow\hat{c}\kappa,\hat{d}\gamma\rangle = U(\lambda\mu\uparrow,\eta)^{\hat{a}}{}_{a}\, U(\eta\nu\uparrow,\gamma)^{\hat{b}}{}_{b} \\
&\qquad \times\langle(\lambda\mu)\uparrow a\eta,\nu,\uparrow b\gamma|\lambda(\mu\nu)\uparrow c\kappa,d\gamma\rangle\, U(\mu\nu\uparrow,\kappa)^{\dagger c}{}_{\hat{c}}\, U(\lambda\kappa\uparrow,\gamma)^{\dagger d}{}_{\hat{d}}\ .
\end{aligned}$$

If the choices of complex conjugation and transposition factors are to be used, the phase restrictions imposed by these choices must also be incorporated in (5.7). If phase freedom remains for a factor, an arbitrary choice can be made within the range specified by the unitary conditions and symmetry conditions of the factor. Having chosen the phase, there is an additional restriction on further phase freedoms. This restriction fixes one of the phase freedom factors relative to the other three appearing in (5.7). In this manner, all phase freedom factors are related to a smaller set, which are termed basis phase freedom factors.

Those factors for ..hich there is no phase freedom are calculated from the symmetry equations (5.3-5), and, in particular, (5.6). The calculation would be performed by initially calculating associative factors of 'lower' group-subgroup schemes, and, from them, building up to associative factors of 'higher' group-subgroup schemes. (We define 'lower' and 'higher' with reference to figure 5.3; $G' \supset H'$ is lower than $G \supset H$, and vice versa). If the group-subgroup schemes are the same, such as for the coupling scheme of the Racah-Wigner algebra, the power of the irrep becomes the key criterion for the recursive method. The power is defined as the smallest integer, p, for which the irrep γ is contained in the Kronecker power $(\varepsilon+\varepsilon^*)^p$, where ε is the primitive (defining) representation of the group. (Butler and Wybourne 1976a).

Figure 5.3



There exists one phase freedom factor for every non primitive irrep. This phase freedom factor is then chosen with respect to the basis phase freedom factors which are labelled at least once by the primitive irrep (see Bickerstaff 1981).

In the next section, we discuss the calculation of the coupling, subduction and induction factors, which is directly connected to the above calculation of associative factors.

6. COUPLING, SUBDUCTION, INDUCTION FACTORS

In this section we derive three identities, one for each of the processes of coupling, subduction and induction. These identities relate an associative factor of a group-subgroup scheme with a lower group-subgroup scheme and appropriate coupling, subduction, or induction factors. We discuss, in connection with these identities, a method of calculation for coupling, subduction, and induction factors.

In any application of the Racah-Wigner coupling algebra to physical calculations via the Wigner-Eckart theorem, it is essential to calculate coupling coefficients and coupling factors. Butler and Wybourne (1976a) has established a systematic methodology for computing 3jm symbols (symmetrised coupling factors) using symmetry equations and the Wigner relation (Butler 1981 eqn. 3.3.29), which relates a 6j symbol (or recoupling factors) of a group with 6j symbols (or recoupling factors) of a subgroup and appropriate 3jm symbols (coupling factors). In analogy to this method, we can calculate

$$\begin{array}{lll} \text{(i) the coupling factors} & \langle\mu\nu b\gamma a\eta|(\mu a_1\lambda;\nu a_2\rho)c\eta\rangle & \\ \text{(ii) the subduction factors} & \langle\gamma a\eta c\lambda\rho|\gamma b\mu(a_1\lambda)\nu(a_2\rho)\rangle & \text{(III.6.1)} \\ \text{(iii) the induction factors} & \langle\mu\nu\uparrow a\gamma b\eta|\rho(\mu a_1\lambda,\nu a_2\rho)\uparrow c\eta\rangle & \end{array}$$

with the corresponding symmetry equations and Wigner relations. The corresponding Wigner relations can be obtained from the three relations (II.2.15, II.2.9, II.5.9) giving the associative factors as a product of four coupling, subduction and induction factors respectively. We do this by

(i) choosing the appropriate group-subgroup bases,

(ii) factorising the 'higher' group-subgroup coupling, induction, or subduction coefficients into corresponding factors and 'lower' group-subgroup coefficients, and

(iii) recombining the lower group-subgroup coefficients to form a 'lower' associative factor.

We give the following results:

$$
\begin{aligned}
\text{(i)}\quad & \langle(\lambda\mu)a\eta,\nu,b\gamma|\lambda(\mu\nu)c\kappa,d\gamma\rangle\\
& \times \langle\lambda\kappa d\gamma b_1\rho|(\lambda a_1\upsilon;\kappa a_2\tau)d'\rho\rangle\\
& \times \langle\mu\nu c\kappa a_2\tau|(\mu a_1\omega;\nu a_2\zeta)c'\tau\rangle\\
& = \langle\eta\nu b\gamma b_1\rho|(\eta b_1\sigma;\nu a_2\zeta)b'\rho\rangle\\
& \times \langle\lambda\mu a\eta b'_1\sigma|(\lambda a_1\upsilon;\mu a'_1\omega)a'\sigma\rangle\\
& \times \langle(\upsilon\omega)a'\sigma,\zeta,b'\rho|\upsilon(\omega\zeta)c'\tau,d'\rho\rangle\ .\\[2ex]
\text{(ii)}\quad & \langle\gamma a\eta(b\lambda\mu)\nu|\gamma c\lambda\kappa(d\mu\nu)\rangle\\
& \times \langle\kappa d\mu(a_1\omega)\nu(a_2\zeta)|\kappa a'_2\tau d'\omega\zeta\rangle\\
& \times \langle\gamma c\lambda(a_1\upsilon)\kappa(a'_2\tau)|\gamma b_1\rho c'\upsilon\tau\rangle\\
& = \langle\eta b\lambda(a'_1\upsilon)\mu(a_1\omega)|\eta b'_1\sigma b'\upsilon\omega\rangle \qquad\qquad \text{(III.6.2)}\\
& \times \langle\gamma a\eta(b'_1\sigma)\nu(a_2\zeta)|\gamma b_1\rho a'\sigma\zeta\rangle\\
& \times \langle\rho a'\sigma(b'\upsilon\omega)\zeta|\rho c'\upsilon\tau(d'\omega\zeta)\rangle\ .\\[2ex]
\text{(iii)}\quad & \langle(\lambda\mu)\uparrow a\eta,\nu,\uparrow b\gamma|\lambda(\mu\nu)\uparrow c\kappa,\uparrow d\gamma\rangle\\
& \times \langle\lambda\kappa\uparrow d\gamma b_1\rho|p_1(\lambda a_1\upsilon;\kappa a_2\tau)\uparrow d'\rho\rangle\\
& \times \langle\mu\nu\uparrow c\kappa a_2\tau|p_2(\mu a'_1\omega;\nu a'_2\zeta)\uparrow c'\tau\rangle\\
& = \langle\eta\nu\uparrow b\gamma b_1\rho|p_3(\eta b'_1\sigma;\nu a'_1\zeta)\uparrow b'\rho\rangle\\
& \times \langle\lambda\mu\uparrow a\eta b'_1\sigma|p_4(\lambda a_1\upsilon;\mu a'_1\omega)\uparrow a'\sigma\rangle\\
& \times \langle(\upsilon\omega)\uparrow a'\sigma,\zeta,\uparrow b'\rho|\upsilon(\omega\zeta)\uparrow c'\tau,\uparrow d'\rho\rangle
\end{aligned}
$$

The calculation of the factors, given in (6.1), would be performed after the associative factors for both 'higher' and 'lower' group-subgroup schemes have been obtained. The calculation proceeds by determining the phase freedom for each of the factors:

$$
\begin{aligned}
&\text{(i)} \quad \langle\mu\nu\hat{b}\gamma\hat{a}\eta|(\mu\hat{a}_1\lambda;\nu\hat{a}_2\rho)\hat{c}\eta\rangle = U(\mu\nu,\gamma)^{\hat{b}}{}_{b}\,U(\gamma,\eta)^{\hat{a}}{}_{a} \\
&\qquad \times\langle\mu\nu b\gamma a\eta|(\mu a_1\lambda;\nu a_2\rho)c\eta\rangle U(\mu,\lambda)^{\dagger a_1}{}_{\hat{a}_1}\;U(\nu,\rho)^{\dagger a_2}{}_{\hat{a}_2}\;U(\lambda\rho,\eta)^{\dagger c}{}_{\hat{c}} \\
&\text{(ii)} \quad \langle\gamma\hat{a}\eta\hat{c}\lambda\rho|\gamma\hat{b}\mu(\hat{a}_1\lambda)\nu(\hat{a}_2\rho)\rangle = U(\gamma,\eta)^{\hat{a}}{}_{a}\,U(\eta,\rho)^{\hat{c}}{}_{c} \qquad \text{(III.6.3)} \\
&\qquad \times\langle\gamma a\eta c\lambda\rho|\gamma b\mu(a_1\lambda)\nu(a_2\rho)\rangle\;U(\gamma,\mu\nu)^{\dagger b}{}_{\hat{b}}\;U(\mu,\lambda)^{\dagger a_1}{}_{\hat{a}_1}\;U(\nu,\rho)^{\dagger a_2}{}_{\hat{a}_2} \\
&\text{(iii)} \quad \langle\mu\nu\uparrow\hat{b}\gamma\hat{a}\eta|(\mu\hat{a}_1\lambda;\nu\hat{a}_2\rho)\uparrow\hat{c}\eta\rangle = U(\mu\nu\uparrow,\gamma)^{\hat{b}}{}_{\hat{b}}\;U(\gamma,\eta)^{\hat{a}}{}_{a} \\
&\qquad \times\langle\mu\nu\uparrow b\gamma a\eta|(\mu a_1\lambda;\nu a_2\rho)\uparrow c\eta\rangle U(\mu,\lambda)^{\dagger a}{}_{\hat{a}_1}\;U(\nu,\rho)^{\dagger a_2}{}_{\hat{a}_2}\;U(\lambda\rho,\eta)^{\dagger c}{}_{\hat{c}}\;.
\end{aligned}
$$

If freedom remains, one makes a choice subject to the restrictions imposed by the previous choices of complex conjugation, transposition, and associative factors, and the symmetry requirements given by (1) the complex conjugation symmetry,

$$
\begin{aligned}
&\text{(i)} \quad \langle\mu^*\nu^*b'\gamma^*a'\eta^*|(\mu^*a_1'\lambda^*;\nu^*a_2'\rho^*)c'\eta^*\rangle = A(\mu\nu,\gamma)^{b'}{}_{b}A(\gamma,\eta)^{a'}{}_{a} \\
&\qquad \times\langle\mu\nu b'\gamma a\eta|(\mu a_1\lambda,\nu a_2\rho)c\eta\rangle^*\;A(\mu,\lambda)^{\dagger a_1^*}{}_{a_1'}\;A(\nu,\rho)^{\dagger a_2^*}{}_{a_2'}\;A(\lambda\rho,\eta)^{\dagger c^*}{}_{c'} \\
&\text{(ii)} \quad \langle\gamma^*a'\eta^*c'\lambda^*\rho^*|\gamma^*b'\mu^*(a_1\lambda^*)\nu^*(a_2'\rho^*)\rangle\;A(\gamma,\eta)^{a'}{}_{a}\;A(\eta,\lambda\rho)^{c'} \\
&\hspace{28em}\text{(III.6.4)} \\
&\qquad \times\langle\gamma a\eta c\lambda\rho|\gamma b\mu(a_1\lambda)\nu(a_2\rho)\rangle^*\;\;A(\gamma,\mu\nu)^{\dagger b^*}{}_{b'}A(\mu,\lambda)^{\dagger a_1^*}{}_{a_1}\;A(\nu,\rho)^{\dagger a_2}{}_{a_2}, \\
&\text{(iii)} \quad \langle\mu^*\nu^*\uparrow b'\gamma^*a'\eta^*|p'(\mu^*a_1'\lambda^*;\nu^*a_2'\rho^*)\uparrow c'\eta^*\rangle = A(\eta\nu\uparrow,\gamma)^{b'}{}_{b}A(\gamma,\eta)^{a'}{}_{a} \\
&\qquad \times\langle\eta\nu\uparrow b\gamma a\eta|p(\mu a_1\lambda;\nu a_2\rho)\uparrow c\eta\rangle^*\;\;A(\mu,\lambda)^{\dagger a_1^*}{}_{a_1}\;A(\nu,\rho)^{\dagger a_2^*}{}_{a_2}\;A(\lambda\rho\uparrow,\eta)^{\dagger c^*}{}_{c'}
\end{aligned}
$$

and (2) the transposition symmetry,

$$
\begin{aligned}
&\text{(i)} \quad \langle\mu\nu b'\gamma a\eta|(\nu a_2\rho;\mu a_2\lambda)c'\eta\rangle = T(\mu\nu,\gamma)^{b'}{}_{b} \\
&\qquad \times\;\langle\mu\nu b\gamma a\eta|(\mu a_1\lambda;\nu a_2\rho)c\eta\rangle\;T(\lambda\rho,\eta)^{\dagger c}{}_{c'}
\end{aligned}
$$

$$\text{(ii)} \quad \langle \gamma a \eta c' \rho \lambda | \gamma b' \nu (a_2 \rho) \mu (a_1 \lambda) \rangle = T(\eta, \lambda\rho)^{c'}{}_{c} \times \langle \gamma a \eta c \lambda \rho | \gamma b \mu (a_1 \lambda) \nu (a_2 \rho) \rangle \, T(\gamma, \mu\nu)^{\dagger b}{}_{b'} \tag{III.6.5}$$

$$\text{(iii)} \quad \langle \nu\mu\uparrow b' \gamma a \eta | p(\nu a_2 \rho ; \mu a_1 \lambda) \uparrow c' \eta \rangle = T(\mu\nu\uparrow, \gamma)^{b'}{}_{b} \times \langle \mu\nu\uparrow b \gamma a \eta | p(\mu a_1 \lambda ; \nu a_2 \rho) \uparrow c \eta \rangle \, T(\lambda\rho\uparrow, \eta)^{\dagger c}{}_{c'} \quad .$$

If no freedom remains the coupling, subduction or induction factor is then calculated using relations (6.2) by choosing a suitable group associative factor. One important consideration which does not arise from any phase freedom argument is the question of the orientation phase choice (Reid and Butler 1982). This orientation choice arises in the 3jm calculation of the point groups and appears to be related to the different embeddings of a subgroup in a group. The exact nature of this choice, as Reid and Butler point out, is unclear.

An example of these methods is given in the next chapter.

"I think the problem is not to find the best or most efficient method to proceed to a discovery, but to find any method at all."

Feynman 1965 in
Nobel Lecture for Physics.

CHAPTER IV

ALGEBRAIC FORMULAE FOR SU_n 6j SYMBOLS

1. INTRODUCTION

In a recent paper (Bickerstaff et al. 1982 see Appendix), we presented computer generated tables of 6j symbols for SU_6 and SU_3, and 3jm factors to power 3 for certain subgroup bases of SU_6 and SU_3. However it is impractical to produce tables for each SU_n group as it arises. The symmetric group/unitary group duality theory, which we shall discuss in Chapters V and VI, shows that many SU_n results are largely n-independent. In this Chapter, we show how to obtain the n-dependence of 6j symbols of SU_n, though we do not use duality theory. We use the building-up method, which we outlined in Section II.5. This method is familiar to nuclear physicists, and has been previously used for SO_3 6j symbols and $SO_3 \supset SO_2$ 3jm symbols (Butler 1976) and the point groups (Butler 1981, Butler and Reid 1979). The building-up method has many advantages over the ladder operator techniques familiar from angular momentum and used by many workers in U_n (see Biedenharn and Louck 1982), and the projection operator technique used for SU_3 by Akiyama and Draayer(1973). A knowledge of character theory, chiefly product and branching rules, is all that is required for the building-up method. This is particularly useful for groups with irreps of large dimension such as E_7 (Butler et al. 1978, 1979) where the other methods are impractical because they involve the construction of very large matrices.

Section 2 gives a summary of the character theory for both SU_n and U_n. We make extensive use of the isomorphism between U_n and the direct product group $U_1 \times SU_n$ to ensure the factorization of U_n 6j symbols into a product of U_1 and SU_n 6j symbols. Section 3 gives a guide to the tables, while section 4 discusses some aspects of the application of the building-up method. The symmetry of the SU_n 6j symbols arising from the transpose-conjugate (or alternating group) symmetry is discussed in section 5. Section 6 discusses symmetries arising from boundary hook removals (and thus the determinantal properties of Schur functions) in the composite (covariant-contravariant) labelling, and some properties of the SU_n symmetry related to the existence of many one dimensional irreps of U_n.

2. U_n and SU_n GROUP INFORMATION

In this section we give a brief outline of the properties of the irreps of U_n and SU_n, and of the ways in which they may be labelled. We have much use for the composite or back-to-back notation used by Murnaghan (1938) and Littlewood (1940) but developed more fully by King and others (Abramsky and King 1970, Black, King and Wybourne 1983).

The irreps of U_n and SU_n, indeed of all compact Lie groups, may be labelled by partitions. A partition of ℓ into p parts, $\lambda_1, \lambda_2, \ldots, \lambda_p$ with $\lambda_1+\lambda_2+\ldots+\lambda_p = \ell$, is denoted $(\lambda_1\lambda_2\ldots\lambda_p)$ or merely λ, and is said to be regular if the parts also satisfy.

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_{p-1} \geqslant \lambda_p > 0. \qquad \text{(IV.2.1)}$$

The character theory of SU_n gives a natural one-to-one correspondence between all regular partitions into at most n-1 parts and the irreps of SU_n. This standard labelling is used in our numerical tables (Bickerstaff et al. 1982). For U_n, a full unique set of standard labels is obtained by using n nonincreasing parts λ, as in (2.1), but relaxing the positivity condition $\lambda_i > 0$. By taking the positive and negative parts as two regular partitions, μ and ν, we obtain the composite label for U_n irreps,

$$\begin{aligned}\{\lambda\} &\equiv \{\lambda_1,\lambda_2 \ldots, \lambda_n\} \\ &= \{\nu_1\nu_2, \ldots \nu_q, 0, \ldots, 0, -\mu_p, \ldots, -\mu_2, -\mu_1\} \\ &= \{\mu;\nu\} \qquad \text{(IV.2.2)}\end{aligned}$$

where $p+q \leqslant n$.

(For U_n and SU_n irreps we enclose the label in braces). The composite label gives an alternative set of labels. The usual association between regular partitions and Young diagrams may be extended to cover those with negative parts by forming the composite diagram. This is obtained by reflecting the Young diagram of μ about the vertical and pairing it back-to-back with the diagram of ν. See King (1970, 1975).

The isomorphism $U_n \simeq U_1 \times SU_n$ gives the relationship between the standard labels of U_n and SU_n.

$$\{\lambda\}(U_n) = \{f\}(U_1) \times \{\lambda'\}(SU_n) \ . \qquad \text{(IV.2.3a)}$$

$$\text{where } f = \sum_{i=1}^{n} \lambda_i \qquad \text{(IV.2.3b)}$$

$$\text{and} \quad \lambda'_i = \lambda_i - \lambda_n \quad \text{for all } i = 1,\ldots,n \qquad \text{(IV.2.3c)}$$

By discarding the U_1 content in (2.3), we can use the composite labels as non-standard labels for SU_n irreps. The prescription for modifying a nonstandard (composite) label for SU_n into a standard SU_n irrep lable is given by (2.3.c).

In the above, all composite labels $\{\mu;\nu\}$ satisfy the condition $p+q \leqslant n$. If the number of parts of $\{\mu;\nu\}$ exceeds n, it forms a nonstandard composite label. This may be standardized by the removal of continuous boundary hooks. The simpler partitions ρ and σ are obtained from μ and ν by removing $(p+q-n-1)$ adjoining cells (a hook) from the lower boundary of the corresponding Young diagrams. The hooks start at the foot of the first column of μ and ν and if they end in columns x and y respectively, then

$$\{\rho;\sigma\} = (-)^{x+y+1}\ \{\mu;\nu\} \quad . \qquad \text{(IV.2.4)}$$

The label vanishes (it labels a null irrep) unless the corresponding Young diagrams of ρ and σ are regular Repeated hook removal may be required to arrive at a standard composite label for U_n. Any composite label of SU_n may be transformed into a standard label by using the hook length removal procedure of (2.4) and then subtracting the n^{th} part as in (2.3c) (see King 1970, 1975).

Using the composite labels for SU_n and U_n, we can express many properties in an almost n-independent manner (Wybourne 1970, Murnaghan 1938, Littlewood 1940, Black, King and Wybourne 1983). For example, the complex conjugate pairs of irreps of U_n are labelled by composite labels with the partitions interchanged, $\{\mu;\nu\}^* = \{\nu;\mu\}$. The Kronecker product rule (Butler and King 1974) for SU_n is given by the prescription

$$\{\mu;\nu\}\times\{\rho;\sigma\} = \sum_{\xi\eta} \{(\mu/\xi).(\rho/\eta);(\nu/\eta).(\sigma/\xi)\} \qquad \text{(IV.2.5)}$$

where "/" and "." are the Schur function operations of division and outer multiplication respectively (tables by P.H. Butler appear in Wybourne 1970). We give an example which will also serve to illustrate the modification rules. For the square of the adjoint (generator) representation, (2.5) gives

$$\{1;1\}\times\{1;1\} = \{2;2\}+\{2;1^2\}+\{1^2;2\}+\{1^2;1^2\}+2\{1;1\}+\{0;0\} \ . \qquad \text{(IV.2.6)}$$

For $n \geqslant 4$, the labels as composite labels are standard. For SU_6 (2.6) becomes

$$\{21^4\}\times\{21^4\}=\{42^4\}+\{3^22^2\}+\{31^3\}+\{2^21^2\}+2\{21^4\}+\{0\} \qquad \text{(IV.2.6)}$$

For $n<4$, one removes boundary hooks as appropriate to give (ϕ the null irrep)

$$SU_3:\{21\}x\{21\}= \{42\}+\{3^2\}+\{3\}+ \phi +2\{21\}+\{0\} \qquad \text{(IV.2.7)}$$

$$SU_2:\{2\}x\{2\} = \{4\}+\phi+\phi-\{2\}+2\{2\}+\{0\}$$

$$= \{4\}+\{2\}+\{0\} \qquad \text{(IV.2.8)}$$

$$SU_1: \ \phi\times\phi = \phi-\phi-\phi-\{0\}+2\phi+\{0\} = \phi \qquad \text{(IV.2.9)}$$

$$SU_0:(-\{0\})\times(-\{0\}) = \phi +\{0\}+\{0\}+\phi -2\{0\}+\{0\} \qquad \text{(IV.2.10)}$$

$$= \{0\} \ .$$

The SU_2 result (2.8) illustrates that the composite labelling gives a natural n dependence to multiplicity separations.

Various formulae have been given for the dimensions of U_n irreps. Table 1 was derived using

$$|\{\mu;\nu\}| = N_n(\mu;\nu) \ / \ H(\mu) \ H(\nu) \ . \qquad \text{(IV.2.11)}$$

The denominator functions $H(\mu)$ and $H(\nu)$ are Robinson's hook formula (Robinson 1961) and $N_n(\mu;\nu)$ may be obtained diagrammatically (El Samra and King 1979).

One important symmetry which we shall be employing is the transpose conjugate or tilde symmetry. This symmetry has its origin in the symmetric group where it is generated by the occurrence of the one dimensional irrep $[1^\ell]$ of S_ℓ. The symmetry relates a pair of S_ℓ irreps $[\tilde{\lambda}]$ and $[\lambda]$ where the partition $\tilde{\lambda}$ is obtained from λ by interchanging rows and columns of the Young diagram. In the application of the transpose conjugate symmetry to the irrep labels of U_n, we emphasise the combinatoric similarities. We have

$$H(\tilde{\mu}) = H(\mu) \qquad \text{(IV.2.12)}$$

and $$N_n(\tilde{\mu};\tilde{\nu}) = (-)^{\ell+m} N_{-n}(\mu;\nu) \qquad \text{(IV.2.13)}$$

More simply, since the dimension formula of $\{\mu;\nu\}$ is a factored polynomial in n with factors (n+a), the dimension formula of $\{\tilde{\mu};\tilde{\nu}\}$ is obtained by replacing all factors (n+a) by (n-a)(see Table 1). This symmetry of the dimension formula also carries through to the Kronecker product rules and into corresponding symmetries of the SU_n 6j results (see section 6).

3. A GUIDE TO THE TABLES

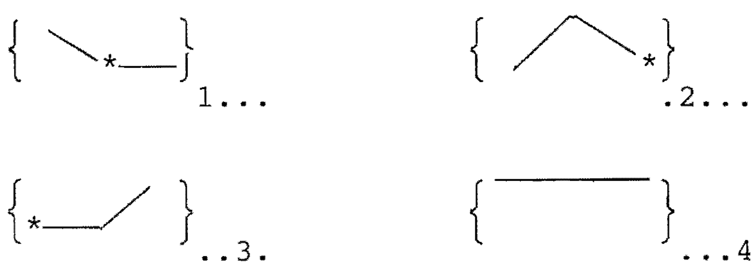
We have listed in Table 1 (p74) the irreps of SU_n up to power 3 giving a composite and standard labellings, complex conjugation properties and algebraic formulae for the dimension of the irreps. The Kronecker product rules are specified by means of triad. A triad is the set of three labels $\gamma_1, \gamma_2, \gamma_3$, together with a multiplicity index r. The triad $(\gamma_1\gamma_2\gamma_3\ r)$ exists if the identity irrep $\{0\}$ occurs at least r-times in the product $\{\gamma_1\}\times\{\gamma_2\}\times\{\gamma_3\}$, or equivalently $\{\gamma_3\}^*$ occurs at least r-times in $\{\gamma_1\}\times\{\gamma_2\}$.

Each irrep has an associated phase, the 1j phase $\{\gamma\}$, which gives the symmetry of the complex conjugation coefficients (see Section III.3) Two (standard) uses of the curly brackets have arisen (i) curly brackets enclosing a partition are used above to denote the irrep, and (ii) in the Racah-Wigner algebra curly brackets denote the various j symbols. But no confusion should arise from the context. We shall omit brackets from the irreps when denoting a j symbol. Each triad has an associated phase, the 3j phase, which gives the symmetry on reordering coupled products. These are denoted $\{\gamma_1\gamma_2\gamma_3\ r\}$ and are listed in Table 2 (p75).

The 6j symbol is related to recouplings between a set of six irreps by means of four couplings. The triads occur in the 6j symbol

$$\begin{Bmatrix} \gamma_1\gamma_2\gamma_3 \\ \eta_1\eta_2\eta_3 \end{Bmatrix}_{r_1r_2r_3r_4} \tag{IV.3.1}$$

in the order



(IV.3.2)

that is, $(\gamma_1\overset{*}{\eta_2}\eta_3r_1)\,(\eta_1\gamma_2\overset{*}{\eta_3}r_2)\ (\overset{*}{\eta_1}\eta_2\gamma_3r_3)\,(\gamma_1\gamma_2\gamma_3r_4)$ respectively. Symmetries are used to reduce the size of the 6j table (see p78). The full symmetries are given in Butler (1981) and Butler and Wybourne (1976), but to find a 6j in the table one needs the following. The 6j symbols are invariant under even permutations of the columns; there is a complex conjugation symmetry

$$\begin{Bmatrix} \gamma_1 & \gamma_2 & \gamma_3 \\ \eta_1 & \eta_2 & \eta_3 \end{Bmatrix}_{r_1r_2r_3r_4} = \begin{Bmatrix} \gamma_1^* & \gamma_2^* & \gamma_3^* \\ \eta_1^* & \eta_2^* & \eta_3^* \end{Bmatrix}^*_{r_1r_2r_3r_4} \quad ; \qquad \text{(IV.3.3)}$$

the row flip symmetries, the (23) flip being

$$= \begin{Bmatrix} \gamma_1^* & \eta_2^* & \eta_3^* \\ \eta_1 & \gamma_2 & \gamma_3^* \end{Bmatrix}_{r_4r_3r_2r_1} \quad ; \qquad \text{(IV.3.4)}$$

the column interchange symmetries, the (12) operation being

$$= \begin{Bmatrix} \gamma_2 & \gamma_1 & \gamma_3 \\ \eta_2^* & \eta_1^* & \eta_3^* \end{Bmatrix}_{r_2r_1r_3r_4} \qquad \text{(IV.3.5.)}$$

$$\times \, \{\eta_1\}\{\eta_2\}\{\eta_3\}\{\gamma_1\eta_2^*\eta_3r_1\}\{\eta_1\gamma_2\eta_3^*r_2\}\{\eta_1^*\eta_2\gamma_3r_3\}\{\gamma_1\gamma_2\gamma_3\gamma_4\}$$

The phase in (3.5) is the same for all interchanges, since we consider irreps of sufficiently small power not to include non-simple phase cases (Butler 1975) such as the irrep {21;21} of SU_n $(n \geqslant 4)$ which is of power 6.

4. THE METHOD OF CALCULATION

The 6j symbols are calculated recursively by building up from the trivial 6j symbols. An outline of this calculation was given in section III.5; detailed accounts with numerical examples have been given by Butler (1976, 1981), Bickerstaff and Wybourne (1981). As with previous examples, simplifications occur in the Racah-Wigner algebra because all unitary groups are quasiambivalent, that is the product of 1j phases $\{\gamma_1\}\{\gamma_2\}\{\gamma_3\} = +1$ if $(\gamma_1\gamma_2\gamma_3 r)$ is a triad (Butler 1975), and no nonsimple phase irreps occur in the present calculations.

The n-dependence of the triads and the dimensions of irreps may be carried through. Also, while we have selected values for some 3j phases we have carried them through as functions. Those 3j phases of the form $\{\mu\mu\lambda r\}$ are given by character theory while others are fixed by the symmetry requirements imposed on the 6j symbols. For example the reality choice of the 6j symbol

$$\begin{Bmatrix} 1;1 & 1^2;0 & 0;1^2 \\ 0;1 & 1;0 & 1;0 \end{Bmatrix}_{0000} = \begin{Bmatrix} 1;1 & 1^2;0 & 0;1^2 \\ 0;1 & 1;0 & 1;0 \end{Bmatrix}^*_{0000} \qquad \text{(IV.4.1)}$$

$$\times \{0;1\}\{1;1\ 1^2;0\ 0;1^2\}\{1;1\ 1;0\ 0;1\}$$

where we have used (3.3) and (3.4), requires that the product of 3j phases appearing on the right is unity. Other 3j choices are given in Table 2. In section 5 an example is given where the transpose conjugate symmetry requires certain further 3j choices. However, all these 3j phase choices can be obtained by requiring that in (3.5) the phase given for the column interchange of a 6j symbol satisfies

$$\{\eta_1\}\{\eta_2\}\{\eta_3\}\{\gamma_1\overset{*}{\eta_2}\gamma_3 r_1\}\{\eta_1\gamma_2\overset{*}{\eta_3}r_2\}\{\overset{*}{\eta_1}\eta_2\gamma_3 r_3\}\{\gamma_1\gamma_2\gamma_3 r_4\}$$

$$= (-)^{r_1+r_2+r_3+r_4} \qquad \text{(IV.4.2)}$$

(Note the maximum multiplicity occurring is two, that is $r=0,1$). For all multiplicity free 6j symbols, the 6j is invariant under column interchange.

5. THE TRANSPOSE CONJUGATE SYMMETRY OF 6j SYMBOLS.

The transpose conjugate symmetry relates two 6j symbols of the form

$$\begin{Bmatrix} \mu_1;\nu_1 & \mu_2;\sigma_2 & \mu_3;\sigma_3 \\ \rho_1;\sigma_1 & \rho_2;\sigma_2 & \rho_3;\sigma_3 \end{Bmatrix}_{r_1r_2r_3r_4} \quad \text{and}$$

$$\begin{Bmatrix} \tilde{\mu}_1;\tilde{\nu}_1 & \tilde{\mu}_2;\tilde{\nu}_2 & \tilde{\mu}_3;\tilde{\nu}_3 \\ \tilde{\rho}_1;\tilde{\sigma}_1 & \tilde{\rho}_2;\tilde{\sigma}_2 & \tilde{\rho}_3;\tilde{\sigma}_3 \end{Bmatrix}_{r_1r_2r_3r_4}$$

The symmetry is traceable to the existence of the one-dimensional alternating irrep $[1^{\ell}]$ of S_{ℓ}. The existence of such irreps allows one to prove that the Schur function operations of multiplication and division are the same for transpose conjugate irreps. (Murnaghan 1938, Littlewood 1940, Robinson 1961). Hence, if $(\mu_1;\nu_1 \;\; \mu_2;\nu_2 \;\; \mu_3;\nu_3 \; r)$ is a triad then so is $(\tilde{\mu}_1;\tilde{\nu}_1 \;\; \tilde{\mu}_2;\tilde{\nu}_2 \;\; \tilde{\mu}_3;\tilde{\nu}_3 \; r)$ providing n is large enough (see 2.5 and Table 2). We have already seen the relationship between the dimensions of $\{\mu;\nu\}$ and $\{\tilde{\mu};\tilde{\nu}\}$. As a consequence of these two similarities, the equations used to solve for transposed conjugate pairs of 6j symbols are closely related. Indeed the 6j symbol obtained from the other by transpose-conjugating the labels is related to it in Table 3 by

(i) replacing all factors n+a by n-a,

(ii) replacing any 3j phase $\{\mu_1;\nu_1 \; \mu_2;\nu_2 \; \mu_3;\nu_3 \; r\}$ by the transpose-conjugate 3j phase $\{\tilde{\mu}_1;\tilde{\nu}_1 \; \tilde{\mu}_2;\tilde{\nu}_2 \; \tilde{\mu}_3;\tilde{\nu}_3 \; r\}$ and,

(iii) replacing any 1j phase $\{\mu;\nu\}$ by $\{\tilde{\mu};\tilde{\nu}\}$.

We note that if a 6j is self transpose conjugate, to within the row and column symmetries discussed in section 3, the n-independent formula for the 6j will reflect this by

(i) having pairs of factors (n+a) (n-a), and

(ii) having self transpose conjugate 3j phases or pairs of 3j phases which are transpose conjugates.

During the calculation of 6j symbols various phase choices arise (see Butler 1975, 1981, Butler and Wybourne 1976, Bickerstaff and Wybourne 1981) and these are chosen so that the 6j symbols satisfy the above transpose conjugate symmetry. Such phases occur for non-basis triads and, in addition, the symmetry imposes conditions on some 3j phases which we would otherwise be free to choose. For example, we have

$$\left\{\begin{matrix} \tilde{1};\tilde{1} & \tilde{0};\tilde{2} & \tilde{1}^2;\tilde{0} \\ \tilde{1};\tilde{0} & \tilde{0};\tilde{1} & \tilde{0};\tilde{1} \end{matrix}\right\}_{0000} = \left\{\begin{matrix} 1;1 & 0;1^2 & 1;0 \\ 1;0 & 0;1 & 0;1 \end{matrix}\right\}_{0000}$$

$$= \left\{\begin{matrix} 1;1 & 0;1 & 1^2;0 \\ 1;0 & 0;1 & 0;1 \end{matrix}\right\}^{*}_{0000} \qquad \text{(IV.5.2)}$$

$$\times \{0;1\}\{1;1\ 1;0\ 0;1\}\{1;0\ 0;1^2\ 1;0\}\{0;1\ 0;1\ 2;0\}\{1;1\ 0;1^2\ 1;0\}$$

after applying a (23) column interchange and a complex conjugation. If the transpose conjugate symmetry is not to introduce additional phase factors then the product of 3j phases in (5.2) must be chosen unity. A further example

$$\left\{\begin{matrix} \tilde{21};\tilde{0} & \tilde{0};\tilde{21} & \tilde{1};\tilde{1} \\ \tilde{2};\tilde{0} & \tilde{1}^2;\tilde{0} & \tilde{0};\tilde{1} \end{matrix}\right\}_{000r} = \left\{\begin{matrix} 21;0 & 0;21 & 1;1 \\ 1^2;0 & 2;0 & 0;1 \end{matrix}\right\}_{r}$$

$$= \left\{\begin{matrix} 21;0 & 0;21 & 1;1 \\ 2;0 & 1^2;0 & 0;1 \end{matrix}\right\}^{*}_{r} \qquad \text{(IV.5.3)}$$

$$\times\{0;2\}\{0;2^2\}\{0;1\}\{21;0\ 0;21\ 1;1\ r\}\{21;0\ 0;2\ 0;1\}\{1^2;0\ 0;21\ 1;0\}\{0;1^2\ 2;0\ 1;1\} .$$

The reality or otherwise of this 6j symbol depends on the symmetry of the 6j

$$\begin{Bmatrix} 21;0 & 0;21 & 1;1 \\ 1;0 & 1;0 & 0;1^2 \end{Bmatrix}_{000r} = \begin{Bmatrix} 21;0 & 0;21 & 1;1 \\ 1;0 & 1;0 & 0;1^2 \end{Bmatrix}^{*}_{0000r} \qquad \text{(IV.5.4)}$$

$$\times\{0;1^2\}\{21;0\ 0;21\ 1;1\ r\}\{1;1\ 1;0\ 0;1\}$$

where we have used (3.3) and (3.5). The product of phases can be chosen

$$\{0;1^2\}\{21;0\ 0;21\ 1;1\ r\}\{1;1\ 1;0\ 0;1\} = (-)^r \qquad \text{(IV.5.5)}$$

which determines the 6j of (5.4), and hence (5.3), to be real for r=0 and imaginary for r=1. The transpose conjugate symmetry requires

$$\{0;21\ 2;0\ 1;0\}\{0;21\ 1^2;0\ 1;0\}\{1;1\ 1;0\ 0;1\}\{0;2\ 1;0\ 1;0\}\{0;1^2\ 1;0\ 1;0\} = +1. \qquad \text{(IV.5.6)}$$

Such choices satisfy the usual SU_2 choice and our previous SU_3 and SU_6 choices (Bickerstaff et al. 1982).

6. THE COMPOSITE LABELLING MODIFICATION SYMMETRY.

The composite labelling system gives rise to several labels for the one irrep. For example, {1;1},{2;0} and {0;2} all label the three dimensional irrep {2} of SU_2. The n independent formulae for the 6j values in Table 3 must reflect this duplicity. To continue the example, the formulae for the 6j symbols

$$\begin{Bmatrix} 1;1 & 1;1 & 1;1 \\ 1;1 & 1;1 & 1;1 \end{Bmatrix}_{0000} \qquad \begin{Bmatrix} 1;1 & 1;1 & 1;1 \\ 0;2 & 0;2 & 0;2 \end{Bmatrix}_{0000} \qquad \text{(IV.6.1)}$$

give the same numerical value for n=2. It is important to recognize that whenever two labels are used for the one irrep, there is only one irrep and therefore only one choice for the corresponding quantities in the Racah-Wigner algebra. Phase choices have been made (for n=2) to ensure the identification above and this choice retained for other values for n when the irreps $\{1;1\}$ and $\{0;2\}$ are distinct. The non-uniqueness of the composite labelling gives rise to natural phase choices for some freedoms.

A partial explanation can be given for vanishings of the n-independent 6j formulae for certain values of n. The vanishing of the label $\{\mu;\nu\}$ for certain n is linked to the n dependence of its dimension. Consider the irreps $\{1;1^2\}$ for various n. The modification rules, the irrep $\{1;1^2\}$ vanishes for n=0,2, while its transpose conjugate $\{1;2\}$ vanishes for n=1. Its dimension must contain factors that vanish for the values n=0,2,-1. Indeed we have

$$|\{1;1^2\}| = (n+1)n(n-2)/2 \quad . \qquad \text{(IV.6.2)}$$

Either the modification rules or this dimension formula may be used to check that these values of n are the only ones for which it vanishes. Many of the n+a factors in the 6j table (for both positive and negative a) may be explained by this argument. In some cases, however, the tabulated result has arisen after cancellation of pairs of such factors.

A more subtle aspect of the modification rule is illustrated by the label $\{1^2;1^2\}$ for n=2,

$$\{1^2;1^2\}(SU_2) = - \{2\}(SU_2) \quad . \qquad \text{(IV.6.3)}$$

This non-standard label occurred in (2.8) and cancelled another term. One triad cancelled another, wherein that case the cancelled triad was associated with a multiplicity. The table presented here is not of sufficient size to understand the consequences for 6j formulae arising from such cancelling triads.

Table 1: Irreps of SU_n.

Composite label	0;0	0;1	1;0	$0;1^2$	$1^2;0$	0;2	2;0
Standard label	0	1	1^{n-1}	1^2	1^{n-2}	2	2^{n-2}
Complex conjugate	0	1;0	0;1	$1^2;0$	$0;1^2$	2;0	0;2
Dimension formula	1	n	n	n(n-1)/2	n(n-1)/2	(n+1)n/2	(n+1)n/2

Composite label	1;1	$0;1^3$	$1^3;0$	0;21	21;0	0;3	3;0
Standard label	21^{n-2}	1^3	1^{n-3}	21	$2^{n-2}1$	3	3^{n-1}
Complex conjugate	1;1	$1^3;0$	$0;1^3$	21;0	0;21	3;0	0;3
Dimension formula	(n+1)(n-1)	n(n-1)(n-2)/6	n(n-1)(n-2)/6	(n+1)n(n-1)/3	(n+1)n(n-1)/3	(n+2)(n+1)n/6	(n+2)(n+1)n/6

Composite label	$1;1^2$	$1^2;1$	1;2	2;1
Standard label	2^21^{n-3}	21^{n-3}	31^{n-2}	32^{n-2}
Complex conjugate	$1^2;1$	$1;1^2$	2;1	1;2
Dimension formula	(n+1)n(n-2)/2	(n+1)n(n-2)/2	(n+2)n(n-1)/2	(n+2)n(n-1)/2

Table 2: SU_n 3j phases and values.

$\{0;0 \quad 0;0 \quad 0;0\} = +1$[a]

$\{1;0 \quad 0;1 \quad 0;0\} = \{0;1\}$[b]

$\{0;1^2 \quad 1;0 \quad 1;0\} = -1$[a]

$\{0;1^2 \quad 1;0 \quad 0;0\} = \{0;1^2\} = +1$[c]

$\{0;2 \quad 1;0 \quad 1;0\} = +1$[a]

$\{2;0 \quad 0;2 \quad 0;0\} = \{0;2\} = +1$[c]

$\{1;1 \quad 1;0 \quad 0;1\} = +1$[d]

$\{1;1 \quad 1^2;0 \quad 0;1^2\} = \{0;1\}\{1;1 \ 1;0 \ 0;1\}$[e]

$\{1;1 \quad 0;2 \quad 1^2;0\} = \{0;1\}\{1;1 \ 1;0 \ 0;1\}\{0;1^2 \ 1;0 \ 1;0\}\{0;2 \ 1;0 \ 1;0\}$[f,g]

$\{1;1 \quad 2;0 \quad 0;2\} = \{0;1\}\{1;1 \ 1;0 \ 0;1\}$[e]

$\{1;1 \quad 1;1 \quad 0;0\} = \{1;1\} = +1$[c]

$\{1;1 \quad 1;1 \quad 1;1r\} = (-)^r\{0;1\}\{1;1 \ 1;0 \ 0;1\}$[c,e,g]

$\{0;1^3 \quad 1^2;0 \quad 1;0\}$

$\{1^3;0 \quad 0;1^3 \quad 0;0\} = \{0;1^3\} = \{0;1\}$[c]

$\{1^3;0 \quad 0;1^3 \quad 1;1\} = \{0;1^2\}\{1;1 \ 1;0 \ 0;1\}$[e]

$\{0;21 \quad 1^2;0 \quad 1;0\}$

$\{0;21 \quad 2;0 \quad 1;0\} = \{0;1^2\}\{0;21 \ 1^2;0 \ 1;0\}\{0;2 \ 1;0 \ 1;0\}\{0;1^2 \ 1;0 \ 1;0\}$[g]

$\{0;21 \quad 1^3;0 \quad 1;1\} = \{0;1^2\}\{0;21 \ 1^2;0 \ 1;0\}\{0;1^3 \ 1^2;0 \ 1;0\}\{1;1 \ 1;0 \ 0;1\}$[g]

$\{21;0 \quad 0;21 \quad 0;0\} = \{0;21\} = \{0.1\}$[c]

$\{21;0 \quad 0;21 \quad 1,1r\} = (-)^{r}\{0;1^2\}\{1;1\ 1;0\ 0;1\}$[e,g]

$\{0;3 \quad 2;0 \quad 1;0\}$

$\{0;3 \quad 21;0 \quad 1;1\} = \{0;2\}\{0;3\ 2;0\ 1;0\}\{0;21\ 2;0\ 1;0\}\{1;1\ 1;0\ 0;1\}$[g]

$\{3;0 \quad 0;3 \quad 0;0\} = \{0;3\} = \{0;1\}$[c]

$\{3;0 \quad 0;3 \quad 1;1\} = \{0;2\}\{1;1\ 1;0\ 0;1\}$[e]

$\{1;1^2 \quad 1^2;0 \quad 0;1\}$

$\{1;1^2 \quad 1;1 \quad 1;0\} = \{1;1^2\ 1^2;0\ 0;1\}\{1;1\ 1;0\ 0;1\}\{0;1^2\ 1;0\ 1;0\}$[g]

$\{1;1^2 \quad 1^3;0 \quad 0;1^2\} = \{1;1^2\ 1^2;0\ 0;1\}\{0;1^3\ 1^2;0\ 1;0\}\{0;1^2\ 1;0\ 1;0\}$[g]

$\{1;1^2 \quad 1^3;0 \quad 0;2\} = \{1;1^2\ 1^2;0\ 0;1\}\{0;1^3\ 1^2;0\ 1;0\ 1;0\}\{0;2\ 1;0\ 1;0\}$[g]

$\{1;1^2 \quad 21;0 \quad 0;1^2\} = \{1;1^2\ 1^2;0\ 0;1\}\{0;21\ 1^2;0\ 1;0\}\{0;1^2\ 1;0\ 1;0\}$[g]

$\{1;1^2 \quad 21;0 \quad 0;2\} = \{1;1^2\ 1^2;0\ 0;1\}\{0;21\ 1^2;0\ 0;1\}\{0;2\ 1;0\ 1;0\}$[g]

$\{1;1^2 \quad 1;1^2 \quad 1^2;0\} = -1$[a]

$\{1;1^2 \quad 1;1^2 \quad 2;0\} = +1$[a]

$\{1^2;1 \quad 1;1^2 \quad 0;0\} = \{1;1^2\}=\{0;1\}$[c]

$\{1^2;1 \quad 1;1^2 \quad 1;1r\} = (-)^{r}\{1;1\}\{1;1\ 1;0\ 0;1\}$[e]

$\{1;2 \quad 2;0 \quad 0;1\}$

$\{1;2 \quad 1;1 \quad 1;0\} = \{1;2\ 2;0\ 0;1\}\{1;1\ 1;0\ 0;1\}\{0;2\ 1;0\ 1;0\}$[g]

$\{1;2 \quad 21;0 \quad 0;1^2\} = \{1;2\ 2;0\ 0;1\}\{0;21\ 2;0\ 1;0\}\{0;1^2\ 1;0\ 1;0\}$[g]

$\{1;2\quad 21;0\quad 0;2\} = \{1;2\ 2;0\ 0;1\}\{0;21\ 2;0\ 1;0\}\{0;2\ 1;0\ 1;0\}$[g]

$\{1;2\quad 3;0\quad 0;1^2\} = \{1;2\ 2;0\ 0;1\}\{0;3\ 2;0\ 1;0\}\{0;1^2\ 1;0\ 1;0\}$[g]

$\{1;2\quad 3;0\quad 0;2\} = \{1;2\ 2;0\ 0;1\}\{0;3\ 2;0\ 1;0\}\{0;2\ 1;0\ 1;0\}$[g]

$\{1;2\quad 1;1^2\quad 1^2;0\} = \{1;2\ 1;1\ 1;0\}\{1;1^2\ 1;1\ 1;0\}\{0;1^2\ 1;0\ 1;0\}$[f,g]

$\{1;2\quad 1;1^2\quad 2;0\} = \{1;2\ 1;1\ 1;0\}\{1;1^2\ 1;1\ 1;0\}\{0;2\ 1;0\ 1;0\}$[f,g]

$\{1;2\quad 1^2;1\quad 1;1\} = \{1;2\ 1;1\ 1;0\}\{1;1^2\ 1;1\ 1;0\}\{1;1\ 1;0\ 0;1\}$[f,g]

$\{1;2\quad 1;2\quad 1^2;0\} = -1$[a]

$\{1;2\quad 1;2\quad 2;0\} = +1$[a]

$\{2;1\quad 1;2\quad 0;0\} = \{1;2\ = \ 0;1\}$[c]

$\{2;1\quad 1;2\quad 1;1r\} = (-)^r\{1;1\}\{1;1\ 1;0\ 0;1\}$[e]

[a] fixed by character theory

[b]

$$\{0;1\} = \begin{cases} \pm 1 \text{ for } SU_{2k+1} \\ \pm 1 \text{ for } SU_{2k} \ k \text{ even} \\ -1 \text{ for } SU_{2k} \ k \text{ odd.} \end{cases}$$

[c] chosen by quasiambivalence.

[d] chosen by modification rules.

[e] fixec by the reality choice of 6j symbol.

[f] fixed by the transpose conjugate symmetry requirement of a 6j symbol.

[g] fixed by a column interchange invariance of a 6j symbol.

Table 3: SU_n 6j formulae

The headings denote the top line of the 6j symbol and each subsequent entry denotes a lower line (three irreps and four multiplicity labels) of a 6j symbol, the interchange sign and the phase and algebraic formula. We have used the following notation.

$$\chi_r^\lambda = \begin{cases} +1 & \text{for } \theta_r^\lambda = +1 \\ +i & \text{for } \theta_r^\lambda = -1 \end{cases} \qquad \delta_r = \tfrac{1}{2}\Big[1+\{1;1\ 1;1\ 1;1\ r\}\Big] = \begin{cases} 1 & \text{if } \{1;1\ 1;1\ 1;1\ r\} = +1 \\ 0 & \text{if } \{1;1\ 1;1\ 1;1\ r\} = -1 \end{cases}$$

$$\theta_r^\lambda = \{\lambda^*\ \lambda\ 1;1\ r\}\{1;1\ 1;0\ 0;1\}\{\mu\} \qquad \text{with } (\lambda\ \mu^*\ 1^*) \text{ forming a triad}$$

$$x = \delta_{r_1}\delta_{r_2}\delta_{r_3} \qquad y = \delta_{r_1}\delta_{r_2} + \delta_{r_2}\delta_{r_3} + \delta_{r_3}\delta_{r_1}$$

$$z = \delta_{r_1} + \delta_{r_2} + \delta_{r_3} \qquad \delta_{rr'}, \text{ the Kronecker delta}$$

0;0	0;0	0;0				
0;0	0;0	0;0	0000	+	+1	+1
1;0	0;1	0;0				
0;0	0;0	0;1	0000	+	{0;1}	$\sqrt{1/n}$
1;0	1;0	0;0	0000	+	{0;1}	$1/n$

$0;1^2$	$1;0$	$1;0$				
$0;0$	$0;1$	$1;0$	0000	+	$\{0;1^2\ 1;0\ 1;0\}$	$1/n$
$1^2;0$	$0;1^2$	$0;0$				
$0;0$	$0;0$	$0;1^2$	0000	+	$\{0;1^2\}$	$\sqrt{2/n(n-1)}$
$1;0$	$1;0$	$0;1$	0000	+	$\{0;1^2\ 1;0\ 1;0\}$	$\sqrt{2}/n\sqrt{n-1}$
$1^2;0$	$1^2;0$	$0;0$	0000	+	$\{0;1^2\}$	$2/n(n-1)$
$0;2$	$1;0$	$1;0$				
$0;0$	$0;1$	$1;0$	0000	+	$\{0;2\ 1;0\ 1;0\}$	$1/n$
$2;0$	$0;2$	$0;0$				
$0;0$	$0;0$	$0;2$	0000	+	$\{0;2\}$	$\sqrt{2/(n+1)n}$
$1;0$	$1;0$	$0;1$	0000	+	$\{0;2\ 1;0\ 1;0\}$	$\sqrt{2}/n\sqrt{n+1}$
$2;0$	$2;0$	$0;0$	0000	+	$\{0;2\}$	$2/(n+1)n$
$1;1$	$1;0$	$0;1$				
$0;0$	$1;0$	$1;0$	0000	+	$\{1;1\ 1;0\ 0;1\}$	$1/n$
$0;1^2$	$0;1$	$0;1$	0000	+	$+1$	$1/n(n-1)$
$0;2$	$0;1$	$0;1$	0000	+	$+1$	$1/(n+1)n$
$1;1$	$1;0$	$1;0$	0000	+	$-\{0;1\}$	$1/(n+1)n(n-1)$

$1;1 \quad 1^2;0 \quad 0;1^2$

$0;0$	$1^2;0$	$1^2;0$	0000	+	$\{1;1 \quad 1^2;0 \quad 0;1^2\}$	$2/n(n-1)$
$0;1$	$1;0$	$1;0$	0000	+	+1	$\sqrt{n-2}/n(n-1)$
$1;1$	$1^2;0$	$1^2;0$	0000	+	+1	$(n-1+\sqrt{5})(n-1-\sqrt{5})/(n+1)n(n-1)(n-2)$

$1;1 \quad 0;2 \quad 1^2;0$

$0;0$	$0;1^2$	$0;2$	0000	+	$\{1;1 \; 0;1 \; 1^2;0\}$	$2/n\sqrt{(n+1)(n-1)}$
$1;0$	$0;1$	$0;1$	0000	+	+1	$\sqrt{1/(n+1)n(n-1)}$
$1;1$	$0;1^2$	$0;1^2$	0000	+	$\{0;1^2 \; 1;0 \; 1;0\}\{0;2 \; 1;0 \; 1;0\}$	$1/(n+1)(n-1)$
$1;1$	$0;1^2$	$0;2$	0000	+	+1	$\sqrt{(n+2)(n-2)}/(n+1)n(n-1)$
$1;1$	$0;2$	$0;1^2$	0000	+	+1	$1/(n+1)(n-1)$

$1;1 \quad 2;0 \quad 0;2$

$0;0$	$2;0$	$2;0$	0000	+	$\{1;1 \; 2;0 \; 0;2\}$	$2/(n+1)n$
$0;1$	$1;0$	$1;0$	0000	+	+1	$\sqrt{n+2}/(n+1)n$
$1;1$	$2;0$	$1^2;0$	0000	+	$\{0;1^2 \; 1;0 \; 1;0\}\{0;2 \; 1;0 \; 1;0\}$	$1/(n+1)(n-1)$
$1;1$	$2;0$	$2;0$	0000	+	+1	$(n+1+\sqrt{5})(n+1-\sqrt{5})/(n+2)(n+1)n(n-1)$

1;1	1;1	0;0				
0;0	0;0	1;1	0000	+	{1;1}	$\sqrt{1/(n+1)(n-1)}$
0;1	0;1	0;1	0000	+	{1;1 1;0 0;1}	$\sqrt{1/(n+1)n(n-1)}$
0;1²	0;1²	0;1²	0000	+	{1;1 1²;0 0;1²}	$\sqrt{2}/(n-1)\sqrt{(n+1)n}$
0;1²	0;1²	0;2	0000	+	{1;1 0;2 1²;0}	$\sqrt{2}/(n-1)\sqrt{(n+1)n}$
0;2	0;2	0;1²	0000	+	{1;1 0;2 1²;0}	$\sqrt{2}/(n+1)\sqrt{n(n-1)}$
0;2	0;2	0;2	0000	+	{1;1 2;0 0;2}	$\sqrt{2}/(n+1)\sqrt{n(n-1)}$
1;1	1;1	0;0	0000	+	{1;1}	$1/(n+1)(n-1)$
1;1	1;1	1;1				
0;1	0;1	0;1	000r	$(-)^r$	χ_r	$\frac{1}{(n+1)(n-1)}\sqrt{\frac{(n+2\delta_r)(n-2\delta_r)}{2n}}$
0;1²	0;1²	0;1²	000r	$(-)^r$	χ_r{0;1² 1;0 1;0}{1;1 1;0 0;1}	$\frac{(n-4\delta_r)}{(n+1)(n-1)}\sqrt{\frac{(n+2\delta_r)}{2n(n-2)(n-2\delta_r)}}$
0;2	0;1²	0;1²	000r	$(-)^r$	χ_r{0;2 1;0 1;0}{1;1 1;0 0;1}	$\frac{1}{(n+1)(n-1)}\sqrt{\frac{(n-2)(n+2\delta_r)}{2n(n-2\delta_r)}}$
0;2	0;2	0;1²	000r	$(-)^r$	χ_r{0;1² 1;0 1;0}{1;1 1;0 0;1}	$\frac{1}{(n+1)(n-1)}\sqrt{\frac{(n+2)(n-2\delta_r)}{2n(n+2\delta_r)}}$

1;1 1;1 1;1

0;2	0;2	0;2	000r	$(-)^{r}$	$\chi_r\{0;2\ 1;0\ 1;0\}\{1;1\ 1;0\ 0;1\}$	$\frac{(n+4\delta_r)}{(n+1)(n-1)}\sqrt{\frac{(n-2\delta_r)}{2(n+2)(n+2\delta_r)n}}$
1;1	1;1	0;0	00rr'	$(-)^{r+r'}$	$\delta_{rr'}\{1;1\ 1;1\ 1;1\ r\}$	$1/(n+1)(n-1)$
1;1	1;1	1;1	$r_1r_2r_3r_4$	$(-)^{\Sigma r_i}$	$\frac{1}{2}[\chi_{r_1}\chi_{r_2}\chi_{r_3}\chi_{r_4}+\chi^*_{r_1}\chi^*_{r_2}\chi^*_{r_3}\chi^*_{4}]$	$\frac{(n+2x)(n-2x)}{2(n+1)(n-1}\ x\ (n^2-4(3x-y+z))\ x \prod_{i=1}^{4}\sqrt{\frac{1}{(n+2\delta_{r_i})(n-2\delta_{r_i})}}$

$0;1^3$ $1^2;0$ 1;0

0;0	0;1	$1^2;0$	0000	+	$\{0;1^3\ 1^2;0\ 1;0\}$	$\sqrt{2}/n\sqrt{n-1}$
0;1	$0;1^2$	1;0	0000	+	$\{0;1^3\}\{0;1^2\ 1;0\ 1;0\}$	$2/n(n-1)$
1;1	0;1	$1^2;0$	0000	+	$\{0;1^3\}\{0;1^3\ 1^2;0\ 1;0\}\{1;1\ 1;0\ 0;1\}$	$2/n(n-1)\sqrt{n-2}$

$1^3;0$ $0;1^3$ 0;0

0;0	0;0	$0;1^3$	0000	+	$\{0;1^3\}$	$\sqrt{6/n(n-1)n-2)}$
1;0	1;0	$0;1^2$	0000	+	$\{0;1^3\ 1^2;0\ 1;0\}$	$\sqrt{6}/n\sqrt{(n-1)(n-2)}$
$1^2;0$	$1^2;0$	0;1	0000	+	$\{0.1^3\ 1^2;0\ 1;0\}$	$2\sqrt{3}/n(n-1)\sqrt{n-2}$
1;1	1;1	$0;1^3$	0000	+	$\{1^3;0\ 0;1^3\ 1;1\}$	$\sqrt{6}/(n-1)\sqrt{(n+1)n(n-2)}$
$1^3;0$	$1^3;0$	0;0	0000	+	$\{0;1^3\}$	$6/n(n-1)(n-2)$

$1^3;0 \quad 0;1^3 \quad 1;1$

$1;0$	$1;0$	$0;1^2$	0000	+	$+1$	$\sqrt{2(n-3)}/n(n-1)\sqrt{n-2}$
$1^2;0$	$1^2;0$	$0;1$	0000	+	$\{1;1\ 1^2;0\ 0;1^2\}\{0;1^2\ 1;0\ 1;0\}$	$2\sqrt{2(n-3)}/n(n-1)(n-2)$
$1;1$	$0;0$	$0;1^3$	0000	+	$\{1^3;0\ 0;1^3\ 1;1\}$	$\sqrt{6}/(n-1)\sqrt{(n+1)n(n-2)}$
$1^3;0$	$1^3;0$	$0;0$	0000	+	$\{1^3;0\ 0;1^3\ 1;1\}$	$6/n(n-1)(n-2)$

$0;21 \quad 1^2;0 \quad 1;0$

$0;0$	$0;1$	$1^2;0$	0000	+	$\{0;21\ 1^2;0\ 1;0\}$	$\sqrt{2}/n\sqrt{n-1}$
$0;1$	$0;1^2$	$1;0$	0000	+	$-\{0;1\}\{0;1^2\ 1;0\ 1;0\}$	$1/n(n-1)$
$1;1$	$0;1$	$1^2;0$	0000	+	$-\{0;1\}\{0;21\ 1^2;0\ 1;0\}\{1;1\ 1;0\ 0;1\}$	$\sqrt{n-2}/(n+1)n(n-1)$

$0;21 \quad 2;0 \quad 1;0$

$0;0$	$0;1$	$2;0$	0000	+	$\{0;21\ 2;0\ 1;0\}$	$\sqrt{2}/n\sqrt{n+1}$
$0;1$	$0;1^2$	$1;0$	0000	+	$+1$	$\sqrt{3}/n\sqrt{(n+1)(n-1)}$
$0;1$	$0;2$	$1;0$	0000	+	$-\{0;1\}\{0;2\ 1;0\ 1;0\}$	$1/(n+1)n$
$1;1$	$0;1$	$1^2;0$	0000	+	$\{0;21\ 1^2;0\ 1;0\}\{1;1\ 1;0\ 0;1\}\{0;1^2\ 1;0\ 1;0\}$	$\sqrt{3}/(n+1)(n-1)\sqrt{n}$
$1;1$	$0;1$	$2;0$	0000	+	$-\{0;1\}\{0;21\ 2;0\ 1;0\}\{1;1\ 1;0\ 0;1\}$	$\sqrt{n+2}/(n+1)n(n-1)$

$0;21\ \ 1^3;0\ \ 1;1$						
$0;0$	$1;1$	$1^3;0$	0000	+	$\{0;21\ 1^3;0\ 1;1\}$	$\sqrt{6}/(n-1)\sqrt{(n+1)n(n-2)}$
$0;1$	$0;1$	$1^2;0$	0000	+	+1	$\sqrt{2}/(n-1)\sqrt{(n+1)n}$
$0;1^2$	$0;1^2$	$1;0$	0000	+	$\{0;21\ 1^2;0\ 1;0\}\{0;1^3\ 1^2;0\ 1;0\}\{0;2\ 1;0\ 1;0\}\{0;1\}$	$\sqrt{2}/(n-1)\sqrt{(n+1)n(n-2)}$
$0;1^2$	$0;2$	$1;0$	0000	+	$\{0;1\}\{0;21\ 1^2;0\ 1;0\}\{0;1^3\ 1^2;0\ 1;0\}\{1;1\ 2;0\ \ 0;1^2\}$	$\sqrt{6}/n(n-1)\sqrt{n+1}$
$21;0\ \ 0;21\ \ 0;0$						
$0;0$	$0;0$	$0;21$	0000	+	$\{0;21\}$	$\sqrt{3/(n+1)n(n-1)}$
$1;0$	$1;0$	$0;1^2$	0000	+	$\{0;21\ 1^2;0\ 1;0\}$	$\sqrt{3}/n\sqrt{(n+1(n-1)}$
$1;0$	$1;0$	$0;2$	0000	+	$\{0;21\ 2;0\ 1;0\}$	$\sqrt{3}/n\sqrt{(n+1)(n-1)}$
$1^2;0$	$1^2;0$	$0;1$	0000	+	$\{0.21\ 1^2;0\ 1;0\}$	$\sqrt{6}/n(n-1)\sqrt{n+1}$
$2;0$	$2;0$	$0;1$	0000	+	$\{0;21\ 2;0\ 1;0\}$	$\sqrt{6}/(n+1)n\sqrt{n-1}$
$1;1$	$1;1$	$0;1^3$	0000	+	$\{0;21\ 1^3;0\ 1;1\}$	$\sqrt{3}/(n+1)(n-1)\sqrt{n}$
$1;1$	$1;1$	$0;21$	rr00	$(-)^{r+r'}$	$\{21;0\ 0;21\ 1;1\ r\}\delta_{rr'}$	$\sqrt{3}/(n+1)(n-1)\sqrt{n}$
$1^3;0$	$1^3;0$	$1;1$	0000	+	$\{0;21\ 1^3;0\ 1;1\}$	$3\sqrt{2}/n(n-1)\sqrt{(n+1)(n-2)}$
$21;0$	$21;0$	$0;0$	0000	+	$\{0;21\}$	$3/(n+1)n(n-1)$

$21;0\quad 0;21\quad 1;1$

$1;0$	$1;0$	$0;1^2$	$000r$	$(-)^r$	$-\{0;21\ 1^2;0\ 1;0\}\{0;2\ 1;0\ 1;0\}\{1;1\ 1;0\ 0;1\}\chi_0^{0;21}$	$1/n(n-1$	
					$\{0;21\ 1^2;0\ 1;0\}\{1;1\ 1;0\ 0;1\}\chi_1^{0;21}$	$\sqrt{(n+2)(n-2)}/(n+1)n(n-1)$	
$1;0$	$1;0$	$0;2$	$000r$	$(-)^r$	$-\{0;21\ 2;0\ 1;0\}\{0;1^2\ 1;0\ 1;0\}\{1;1\ 1;0\ 0;1\}\chi_0^{0;21}$	$1/(n+1)n$	for $r=0$
					$\{0;21\ 2;0\ 1;0\}\{1;1\ 1;0\ 0;1\}\chi_1^{0;21}$	$\sqrt{(n+2)(n-2)}/(n+1)n(n-1)$	for $r=1$
$1^2;0$	$1^2;0$	$0;1$	$000r$	$(-)^r$	$-\{0;21\ 1^2;0\ 1;0\}\{0;1\}\chi_0^{0;21}$	$\sqrt{n-2}/(n+1)n(n-1)$	for $r=0$
					$\{0;21\ 1^2;0\ 1;0\}\{0;1^2\ 1;0\ 1;0\}\{0;1\}\chi_1^{0;21}$	$2\sqrt{n+2}/(n+1)n(n-1)$	for $r=1$
$2;0$	$1^2;0$	$0;1$	$000r$	$(-)^r$	$\{0;21\ 2;0\ 1;0\}\{0;2\ 1;0\ 1;0\}\chi_0^{0;21}$	$\sqrt{3}/(n+1)(n-1)\sqrt{n}$	for $r=0$
						0	for $r=1$
$2;0$	$2;0$	$0;1$	$000r$	$(-)^r$	$-\{0;21\ 2;0\ 1;0\}\{0;1\}\chi_0^{0;21}$	$\sqrt{n+2}/(n+1)(n-1)\sqrt{n}$	for $r=0$
					$\{0;21\ 2;0\ 1;0\}\{1;1\ 1;0\ 0;1\}\chi_1^{0;21}$	$2\sqrt{n-2}/(n+1)n(n-1)$	for $r=1$
$1;1$	$0;0$	$0;21$	$0r0r'$	$(-)^{r+r'}$	$\{21;0\ 0;21\ 1;1\ \ r\}\delta_{rr'}$	$\sqrt{3}/(n+1)(n-1)\sqrt{n}$	
$21;0$	$21;0$	$0;0$	$00rr'$	$(-)^{r+r'}$	$\{21;0\ 0;21\ 1;1\ \ r\}\delta_{rr'}$	$3/(n+1)n(n-1)$	

$0;3\quad 2;0\quad 1;0$

$0;0$	$0;1$	$2;0$	0000	$+$	$\{0;3\ 2;0\ 1;0\}$	$\sqrt{2}/n\sqrt{n+1}$
$0;1$	$0;2$	$1;0$	0000	$+$	$\{0;3\}\{0;2\ 1;0\ 1;0\}$	$2/(n+1)n$
$1;1$	$0;1$	$2;0$	0000	$+$	$\{0;1\}\{0;3\ 2;0\ 1;0\}\{1;1\ 1;0\ 0;1\}$	$2/(n+1)n\sqrt{n+2}$

0;3 21;0 1;1

0;0	1;1	21;0	0000	+	{0;3 21;0 1;1}	$\sqrt{3}/(n+1)(n-1)\sqrt{n}$
0;1	0;1	2;0	0000	+	{0;3 21;0 1;1}{0;3 2;0 1;0}{0;21 2;0 1;0}	$\sqrt{2}/(n+1)\sqrt{n(n-1)}$
$0;1^2$	0;2	1;0	0000	+	{0;3 21;0 1;1}{0;21 2;0 1;0}{0;21 1^2;0 1;0}	$\sqrt{6}/(n+1)n\sqrt{n-1}$
0;2	0;2	1;0	0000	+	−{0;3 21;0 1;1}{1;1 2;0 0;2}	$\sqrt{2}/(n+1)\sqrt{(n+2)n(n-1)}$

3;0 0;3 0;0

0;0	0;0	0;3	0000	+	{0;3}	$\sqrt{6/(n+2)(n+1)n}$
1;0	1;0	0;2	0000	+	{0;3 2;0 1;0}	$\sqrt{6}/n\sqrt{(n+2)(n+1)}$
2;0	2;0	0;1	0000	+	{0;3 2;0 1;0}	$2\sqrt{3}/(n+1)n\sqrt{n+2}$
1;1	1;1	0;21	0000	+	{0;3 21;0 1;1}	$\sqrt{6}/(n+1)\sqrt{(n+2)n(n-1)}$
1;1	1;1	0;3	0000	+	{3;0 0;3 1;1}	$\sqrt{6}/(n+1)\sqrt{(n+2)n(n-1)}$
21;0	21;0	1;1	0000	+	{0;3 21;0 1;1}	$3\sqrt{2}/(n+1)n\sqrt{(n+2)(n-1)}$
3;0	3;0	0;0	0000	+	{0;3}	$6/(n+2)(n+1)n$

3;0 0;3 1;1

1;0	1;0	0;2	0000	+	+1	$\sqrt{2(n+3)}/(n+1)n\sqrt{n+2}$
2;0	2;0	0;1	0000	+	{1;1 2;0 0;2}{0;2 1;0 1;0}	$2\sqrt{2(n+3)}/(n+2)(n+1)n$
1;1	0;0	0;3	0000	+	{3;0 0;3 1;1}	$\sqrt{6}/(n+1)\sqrt{(n+2)n(n-1)}$
3;0	3;0	0;0	0000	+	{3;0 0;3 1;1}	$6/(n+2)(n+1)n$

$1;1^2 \quad 1^2;0 \quad 0;1$

$0;0$	$1;0$	$1^2;0$	0000	$+$	$\{1;1^2\ 1^2;0\ 0;1\}$	$\sqrt{2}/n\sqrt{n-1}$
$1;1$	$1;0$	$1^2;0$	0000	$+$	$-\{1;1^2\ 1^2;0\ 0;1\}\{0;1^2\ 1;0\ 1;0\}$	$2/(n+1)n(n-1)\sqrt{n-2}$
$0;1^3$	$0;1^2$	$0;1$	0000	$+$	$\{1;1^2\}$	$2/n(n-1)(n-2)$
$0;21$	$0;1^2$	$0;1$	0000	$+$	$\{1;1^2\}$	$2/(n+1)n(n-1)$

$1;1^2 \quad 1;1 \quad 1;0$

$0;0$	$0;1$	$1;1$	0000	$+$	$\{1;1^2\ 1;1\ 1;0\}$	$\sqrt{1/(n+1)n)n-1)}$
$0;1$	$0;1^2$	$0;1$	0000	$+$	$+1$	$\sqrt{2}/(n-1)\sqrt{(n+1)n}$
$1;0$	$1;1$	$1;0$	0000	$+$	$\{1;1^2\}\{0;1^2\ 1;0\ 1;0\}$	$1/(n+1)(n-1)$
$1^2;0$	$1;0$	$1^2;0$	0000	$+$	$\{1;1^2\ 1^2;0\ 0;1\}$	$\sqrt{2}/(n-1)\sqrt{(n+1)n(n-2)}$
$2;0$	$1;0$	$1^2;0$	0000	$+$	$-\{1;1^2\ 1^2;0\ 0;1\}$	$\sqrt{2}/(n+1)n\sqrt{n-1}$
$1;1$	$0;1$	$1;1$	$0r00$	$(-)^r$	$\chi_r\{0;1\}\{1;1^2\ 1;1\ 1;0\}\{0;1^2\ 1;0\ 1;0\}$	$\sqrt{n+2\delta r}/(n+1)(n-1)\sqrt{2n(n-2\delta r)}$
$1;1^2$	$0;1^2$	$0;1$	0000	$+$	-1	$2/(n+1)n(n-1)(n-2)$
$1^2;1$	$1;1$	$1;0$	0000	$+$	$\{1;1^2\}$	$1/(n+1)(n-1)(n-2)$

$1;1^2 \quad 1^3;0 \quad 0;1^2$

$0;0$	$1^2;0$	$1^3;0$	0000	+	$\{1;1^2\ 1^3;0\ 0;1^2\}$	$\sqrt{6}/n(n-1)\sqrt{n-2})$
$0;1$	$1;0$	$1^2;0$	0000	+	$+1$	$\sqrt{2(n-3}/n(n-1)\sqrt{n-2}$
$0;1^2$	$1;1$	$1;0$	0000	+	$\{1;1^2\ 1^2;0\ 0;1\}\{0;1^3\ 1^2;0\ 1;0\}\{1;1\ 1;0\ 0;1\}$	$2\sqrt{n-3}/(n-1)(n-2)\sqrt{(n+1)n}$
$0;1^3$	$0;1$	$1;1$	0000	+	$\{0;1\}\{1;1\ 1;0\ 0;1\}$	$2\sqrt{2}/(n-1)(n-2)\sqrt{(n+1)n}$
$0;21$	$0;1$	$1;1$	0000	+	$\{0;1\}\{1;1\ 1;0\ 0;1\}$	$\sqrt{2(n-3)}/(n+1)n(n-1)\sqrt{n-2}$
$1;1^2$	$1;0$	$1^2;0$	0000	+	$-\{0;1\}\{0;1^2\ 1;0\ 1;0\}$	$4/(n+1)n(n-1)(n-2)$

$1;1^2 \quad 1^3;0 \quad 0;2$

$0;0$	$2;0$	$1^3;0$	0000	+	$\{1;1^2\ 1^3;0\ 0;2\}$	$2\sqrt{3}/n\sqrt{(n+1)(n-1)(n-2)}$
$0;1$	$1;0$	$1^2;0$	0000	+	$+1$	$\sqrt{2}/n\sqrt{(n+1(n-2)}$
$0;1^2$	$1;1$	$1;0$	0000	+	$\{1;1^2\ 1^2;0\ 0;1\}\{0;1^3\ 1^2;0\ 1;0\}\{1;1\ 1;0\ 0;1\}$	$2/n\sqrt{(n+1)(n-1)(n-2)}$
$0;21$	$0;1$	$1;1$	0000	+	$\{0;1^2\ 1;0\ 1;0\}$	$\sqrt{6}/(n+1)n\sqrt{(n-1)(n-2)}$

$1;1^2 \quad 21;0 \quad 0;1^2$

$0;0$	$1^2;0$	$21;0$	0000	+	$\{1;1^2\ 21;0\ 0;1^2\}$	$\sqrt{6}/n(n-1)\sqrt{n-2}$
$0;1$	$1;0$	$1^2;0$	0000	+	$+1$	$\sqrt{2}/(n-1)\sqrt{(n+1)n}$
$0;1^2$	$1;1$	$1;0$	0000	+	$-\{1;1^2\ 1^2;0\ 0;1\}\{0;21\ 1^2;0\ 1;0\}\{1;1\ 1;0\ 0;1\}$	$1/(n+1)(n-1)\sqrt{n-2}$
$0;2$	$1;1$	$1;0$	0000	+	$\{0;1\}\{1;1^2\ 1^2;0\ 0;1\}\{0;21\ 2;0\ 1;0\}\{0;1^2\ 1;0\ 1;0\}$	$\sqrt{3}/(n+1)(n-1)\sqrt{n}$
$0;1^3$	$0;1$	$1;1$	0000	+	$-\{0;1\}$	$\sqrt{2}/(n-1)(n-2)\sqrt{(n+1)n}$

$1;1^2 \quad 21;0 \quad 0;1^2$

$0;21$	$0;1$	$1;1$	$0r00$	$(-)^r$	$-\chi_0^{0;21}\{0;21\ 1^2;0\ 1;0\}\{0;2\ 1;0\ 1;0\}\{0;1\}$	$2/(n+1)n(n-1)$	for $r=0$
					$\chi_1^{0;21}\{0;21\ 1^2;0\ 1;0\}\{0;2\ 1;0\ 1;0\}\{0;1^2\ 1;0\ 1;0\}\{0;1\}$	$\sqrt{n+2}/(n+1)n(n-1)(n-2)$	for $r=1$
$1;1^2$	$1;0$	$1^2;0$	0000	$+$	$\{0;1\}\{0;1^2\ 1;0\ 1;0\}$	$2/(n+1)n(n-1)(n-2)$	

$1;1^2 \quad 21;0 \quad 0;2$

$0;0$	$2;0$	$21;0$	0000	$+$	$\{1;1^2\ 21;0\ 0;2\}$	$\sqrt{6}/(n+1)n\sqrt{n-1}$	
$0;1$	$1;0$	$1^2;0$	0000	$+$	$+1$	$\sqrt{2(n+2)}/(n+1)n\sqrt{n-1}$	
$0;1^2$	$1;1$	$1;0$	0000	$+$	$\{1;1^2\ 1^2;0\ 0;1\}\{0;21\ 1^2;0\ 1;0\}$	$\sqrt{n+2}/(n+1)n(n-1)$	
$0;2$	$1;1$	$1;0$	0000	$+$	$\{0;1\}\{1;1^2\ 1^2;0\ 0;1\}\{0;21\ 1^2;0\ 1;0\}\{0;1^2\ 1;0\ 1;0\}$	$\sqrt{3}/(n+1)(n-1)\sqrt{n}$	
$0;21$	$0;1$	$1;1$	$0r00$	$(-)^r$	$\chi_0^{0;21}\{0;21\ 1^2;0\ 1;0\}$	$\sqrt{3}/(n+1)(n-1)\sqrt{(n+2)n}$	$r=0$
					$\chi_1^{0;21}\{0;21\ 1^2;0\ 1;0\}$	$\sqrt{3}/(n+1)(n-1)\sqrt{n(n-2)}$	$r=1$
$0;3$	$0;1$	$1;1$	0000	$+$	$-\{0;1^2\ 1;0\ 1;0\}$	$\sqrt{6}/(n+1)n\sqrt{(n+2)(n-1)}$	

$1;1^2 \quad 1;1^2 \quad 1^2;0$

$1;0$	$0;1$	$1;1$	0000	$+$	$+1$	$\sqrt{n-3}/(n-1)\sqrt{(n+1)n(n-2)}$
$1^2;0$	$0;0$	$1^2;1$	0000	$+$	$\{1;1^2\ 1;1^2\ 1^2;0\}$	$2/n\sqrt{(n+1)(n-1)(n-2)}$
$1;1$	$0;1^2$	$0;1$	0000	$+$	$\{0;1^2\ 1;0\ 1;0\}$	$\sqrt{2(n-3)}/(n+1)(n-1)(n-2)$
$1^3;0$	$1;0$	$1^2;0$	0000	$+$	$\{1;1^2\ 1;1\ 1;0\}\{1;1^2\ 1^2;0\ 0;1\}$	$2\sqrt{2}/(n-1)(n-2)\sqrt{(n+1)n}$

$1;1^2$ $1;1^2$ $1^2;0$						
$21;0$	$1;0$	$1^2;0$	0000	+	$-\{1;1^2\ 1;1\ 1;0\}\{1;1^2\ 1^2;0\ 0;1\}$	$\sqrt{2(n-3)}/(n+1)n(n-1)\sqrt{n-2}$
$1^2;1$	$0;1$	$1;1$	0r00	$(-)^r$	$\chi_0^{1;1^2}\{0;1\}$	$n\sqrt{2(n-3)}/(n+1)(n-1)(n-2)\sqrt{n^3-2n^2-2n+2}$ for $r=0$
					$\chi_1^{1;1^2}$	$2\sqrt{n+2}/(n+1)n(n-2)\sqrt{n^3-2n^2-2n+2}$ for $r=1$
$1^2;1$	$1;1^2$	$0;0$	0000	+	$\{1;1^2\ 1;1^2\ 1^2;0\}$	$2/(n+1)n(n-2)$
$1;1^2$	$1;1^2$	$2;0$				
$1;0$	$0;1$	$1;1$	0000	+	$+1$	$1/(n+1)\sqrt{n(n-2)}$
$2;0$	$0;0$	$1^2;1$	0000	+	$\{1;1^2\ 1;1^2\ 2;0\}$	$2/(n+1)n\sqrt{n-2}$
$1;1$	$0;1^2$	$0;1$	0000	+	$\{0;2\ 1;0\ 1;0\}$	$\sqrt{2}/(n+1)\sqrt{n(n-1)(n-2)}$
$21;0$	$1;0$	$1^2;0$	0000	+	$-\{0;1\}$	$\sqrt{6}/(n+1)n\sqrt{(n-1)(n-2)}$
$1^2;1$	$1;1^2$	$0;0$	0000	+	$\{1;1^2\ 1;1^2\ 2;0\}$	$2/(n+1)n(n-2)$
$1^2;1$ $1;1^2$ $0;0$						
$0;0$	$0;0$	$1;1^2$	0000	+	$\{1;1^2\}$	$\sqrt{2/(n+1)n(n-2)}$
$0;1$	$0;1$	$0;1^2$	0000	+	$\{1;1^2\ 1^2;0\ 1;0\}$	$\sqrt{2}/n\sqrt{(n+1)(n-2)}$
$1;0$	$1;0$	$1;1$	0000	+	$\{1;1^2\ 1;1\ 1;0\}$	$\sqrt{2}/n\sqrt{(n+1(n-2)}$
$0;1^2$	$0;1^2$	$0;1^3$	0000	+	$\{1;1^2\ 1^3;0\ 0;1^2\}$	$2/n\sqrt{(n+1)(n-1)(n-2)}$
$0;1^2$	$0;1^2$	$0;21$	0000	+	$\{1;1^2\ 21;0\ 0;1^2\}$	$2/n\sqrt{(n+1)(n-1)(n-2)}$

$1^2;1 \quad 1;1^2 \quad 0;0$

$1^2;0$	$1^2;0$	$1;0$	0000	+	$\{1;1^2\ 1^2;0\ 0;1\}$	$2/n\sqrt{(n+1)(n-1)(n-2)}$
$1^2;0$	$1^2;0$	$1^2;1$	0000	+	$\{1;1^2\ 1;1^2\ 1^2;0\}$	$2/n\sqrt{(n+1)(n-1)(n-2)}$
$0;2$	$0;2$	$0;1^3$	0000	+	$\{1;1^2\ 1^3;0\ 0;2\}$	$2/(n+1)n\sqrt{n-2}$
$0;2$	$0;2$	$0;21$	0000	+	$\{1;1^2\ 21;0\ 0;2\}$	$2/(n+1)n\sqrt{n-2}$
$2;0$	$2;0$	$1^2;1$	0000	+	$\{1;1^2\ 1;1^2\ 2;0\}$	$2/(n+1)n\sqrt{n-2}$
$1;1$	$1;1$	$0;1$	0000	+	$\{1;1^2\ 1;1\ 1;0\}$	$\sqrt{2}/(n+1)\sqrt{n(n-1)(n-2)}$
$1;1$	$1;1$	$1;1^2$	rr'00	$(-)^{r+r'}$	$\{1^2;1\ 1;1^2\ 1;1\ r\}\delta_{rr'}$	$\sqrt{2}/(n+1)\sqrt{n(n-1)(n-2)}$
$1^3;0$	$1^3;0$	$1^2;0$	0000	+	$\{1;1^2\ 1^3;0\ 0;1^2\}$	$2\sqrt{3}/n(n-2)\sqrt{(n+1)(n-1)}$
$1^3;0$	$1^3;0$	$2;0$	0000	+	$\{1;1^2\ 1^3;0\ 0;2\}$	$2\sqrt{3}/n(n-2)\sqrt{(n+1)(n-1)}$
$21;0$	$21;0$	$1^2;0$	0000	+	$\{1;1^2\ 21;0\ 0;1^2\}$	$\sqrt{6}/(n+1)n\sqrt{(n-1)(n-2)}$
$21;0$	$21;0$	$2;0$	0000	+	$\{1;1^2\ 21;0\ 0;2\}$	$\sqrt{6}/(n+1)n\sqrt{(n-1)(n-2)}$
$1;1^2$	$1;1^2$	$0;1^2$	0000	+	$\{1;1^2\ 1;1^2\ 1^2;0\}$	$2/(n+1)n(n-2)$
$1;1^2$	$1;1^2$	$0;2$	0000	+	$\{2;2^2\ 2;1^2\ 2;0\}$	$2/(n+1)n(n-2)$
$1^2;1$	$1^2;1$	$0;0$	0000	+	$\{1;1^2\}$	$2/(n+1)n(n-2)$
$1^2;1$	$1^2;1$	$1;1$	rr'00	$(-)^{r+r'}$	$\{1^2;1\ 1;1^2\ 1;1\ r\}\delta_{rr'}$	$2/(n+1)n(n-2)$

$1^2;1 \quad 1;1^2 \quad 1;1$

$0;1$	$0;1$	$0;1^2$	$000r$	$(-)^r$	$\chi_0^{1;1^2}\{0;1\}\{1;1\ 1;0\ 0;1\}$	$2(2n-1)/(n+1)n(n-1)\sqrt{(n-2)(n^3-2n^2-2n+2}$	for $r=0$
					$\chi_1^{1;1^2}$	$\sqrt{2(n+2)(n-3)}/(n+1)\sqrt{(n-2)(n^3-2n^2-2n+2}$	for $r=1$
$1;0$	$1;0$	$1;1$	$000r$	$(-)^r$	$\chi_0^{1;1^2}$	$\sqrt{n^3-2n^2-2n+2}/(n+1)n(n-1)\sqrt{n-2}$	for $r=0$
						0	for $r=1$
$1^2;0$	$1^2;0$	$1;0$	$000r$	$(-)^r$	$\chi_0^{1\ 1^2}\{0;1\}\{1;1\ 1;0\ 0;1\}$	$2\sqrt{n^3-2n^2-2n+2}/(n+1)n(n-1)(n-2)$	for $r=0$
						0	for $r=1$
$1;1$	$0;0$	$1;1^2$	$0r0r'$	$(-)^{r+r'}$	$\{1^2;1\ 1;1^2\ 1;1r\}\delta_{rr'}$	$\sqrt{2}/(n+1)\sqrt{n(n-1)(n-2)}$	
$1;1$	$1;1$	$0;1$	$00rr'$		$\chi_r^{1;1}\chi_0^{1;1^2}\{1;1\ 1;1\ 1;1\ r\}$	$(n^2-2(1+\delta r)n+2)/(n+1)(n-1)$ $\times\ \sqrt{(n+2\delta r)/2n(n-2\delta r)(n-2)(n^3-2n^2-2n+2)}$	for $r=0$
					$\chi_r^{1;1}\chi_1^{1;1^2}\{1;1\ 1;1\ 1;1\ r\}$	$(n+2-2\delta r)/(n+1)$ $\times\ \sqrt{(n+2\delta r)(n-3)/(n+2)n(n-2\delta r)(n-2)(n^3-2n^2-2n+2)}$	for $r=1$
$1;1^2$	$0;1$	$0;1^2$	$000r$	$(-)^r$	$\chi_0^{1\ 1^2}\{0;1\}$	$n\sqrt{2(n-3)}/(n+1)(n-1)(n-2)\sqrt{n^3-2n^2-2n+2}$	for $r=0$
					$-\chi_1^{1;1^2}$	$2\sqrt{n+2}/(n+1)n(n-2)\sqrt{n^3-2n^2-2n-2}$	for $r=1$
$1^2;1$	$1;0$	$1;1$	$0r0r'$	$(-)^{r+r'}$	$\chi_0^{1;1^2}\chi_0^{1;1^2}\{1^2;1\ 1;1^2\ 1;1\ 0\}$	$(n^2-6n+4)/(n+1)(n-1)(n-2)/n^3-2n^2-2n+2)$	for $r=0$ $r'=0$
					$\chi_0^{1\ 1^2}\chi_1^{1\ 1^2}\{1^2;1\ 1;1^2\ 1;1\ 0\}$	$(n-1)\sqrt{2(n+2)(n-3)}/(n+1)(n-2)(n^3-2n^2-2n+2)$	for $r=0$ $r'=1$
					$\chi^{1\ 1^2}\chi^{1\ 1^2}\{1^2;1\ 1;1^2\ 1;1\ 0\}\{0;1\}$	$2(2n-1)/(n+1)n(n-2)(n^3-2n^2-2n+2)$	for $r=1$ $r'=1$

1;2   2;0   0;1

0;0	1;0	2;0	0000	+	{1;2 2;0 0;1}	$\sqrt{2}/n\sqrt{n+1}$
1;1	1;0	2;0	0000	+	−{1;2 2;0 0;1}{0;2 1;0 1;0}	$2/(n+1)n(n-1)\sqrt{n+2}$
0;21	0;2	0;1	0000	+	{1;2}	$2/(n+1)n(n-1)$
0;3	0;2	0;1	0000	+	{1;2}	$2/(n+2)(n+1)n$

1;2   1;1   1;0

0;0	0;1	1;1	0000	+	{1;2 1;1 1;0}	$\sqrt{1/(n+1)n(n-1)}$
0;1	0;2	0;1	0000	+	+1	$\sqrt{2}/(n+1)\sqrt{n(n-1)}$
1;0	1;1	1;0	0000	+	{1;2}{0;2 1;0 1;0}	$1/(n+1)(n-1)$
$1^2;0$	1;0	2;0	0000	+	−{1;2 2;0 0;1}	$\sqrt{2}/n(n-1)\sqrt{n+1}$
2;0	1;0	2;0	0000	+	{1;2 2;0 0;1}	$\sqrt{2}/(n+1)\sqrt{(n+2)n(n-1)}$
1;1	0;1	1;1	0r00	$(-)^r$	$\chi_r\{0;1\}\{1;2\ 1;1\ 1;0\}\{0;2\ 1;0\ 1;0\}$	$\sqrt{n-2\delta_r}/(n+1)(n-1)\sqrt{2(n+2\delta_r)n}$
$1^2;1$	1;1	1;0	0000	+	{1;2}	$1/(n+1)n(n-1)$
1;2	0;2	0;1	0000	+	−1	$2/(n+2)(n+1)n(n-1)$
2;1	1;1	1;0	0000	+	{1;2}	$1/(n+2)(n+1)(n-1)$

1;2 21;0 0;1²						
0;0	1²;0	21;0	0000	+	{1;2 21;0 0;1²}	$\sqrt{6}/n(n-1)\sqrt{n+1}$
0;1	1;0	2;0	0000	+	+1	$\sqrt{2(n-2)}/n(n-1)\sqrt{n+1}$
0;1²	1;1	1;0	0000	+	{1;2 2;0 0;1}{0;21 1²;0 1;0}{1;1 1²;0 0;1²}{0;1² 1;0 1;0}	$\sqrt{3}/(n+1)(n-1)\sqrt{n}$
0;2	1;1	1;0	0000	+	{1;2 2;0 0;1}{0;21 2;0 1;0}{1;1 1;0 0;1}	$\sqrt{n-2}/(n+1)n(n-1)$
0;1³	0;1	1;1	0000	+	-{0;2 1;0 1;0}	$\sqrt{6}/n(n-1)\sqrt{(n+1)(n-2)}$
0;21	0;1	1;1	0r00	$(-)^r$	$\chi_0^{0;21}${0;21 1²;0 1;0}	$\sqrt{3}/(n+1)(n-1)\sqrt{n(n-2)}$ for r=0
					$\chi^{0;21}${0;21 1²;0 1;0}	$\sqrt{3}/(n+1)(n-1)\sqrt{(n+2)n}$ for r=1
1;1²	1;0	2;0	0000	+	-1	$2\sqrt{3}/(n+1)n(n-1)\sqrt{(n+2)n-2)}$
1;2 21;0 0;2						
0;0	2;0	21;0	0000	+	{1;2 21;0 0;2}	$\sqrt{6}/(n+1)n\sqrt{n-1}$
0;1	1;0	2;0	0000	+	+1	$\sqrt{2}/(n+1)\sqrt{n(n-1)}$
0;1²	1;1	1;0	0000	+	{0;1}{1;2 2;0 0;1}{0;21 1²;0 1;0}{0;2 1;0 1;0}	$\sqrt{3}/(n+1(n-1)\sqrt{n}$
0;2	1;1	1;0	0000	+	-{1;2 2;0 0;1}{0;21 2;0 1;0}{1;1 1;0 0;1}	$1/(n+1)(n-1)\sqrt{n+2}$
0;21	0;1	1;1	0r00	$(-)^r$	$-\chi^{0;21}${0;21 2;0 1;0}{0;1² 1;0 1;0}{0;1}	$2/(n+1)n(n-1)$
					$\chi_1^{0;21}${0;21 2;0 1;0}{0;2 1;0 1;0}{0;1² 1;0 1;0}{0;1}	$\sqrt{n-2}/(n+1)n(n-1)\sqrt{n+2}$
0;1³	0;1	1;1	0000	+	-{0;1}	$\sqrt{2}/(n+2)(n+1)\sqrt{n(n-1)}$
1;2	1;0	2;0	0000	+	{0;1}{0;2 1;0 1;0}	$2/(n+2)(n+1)n(n-1)$

$1;2\ \ 3;0\ \ 0;1^2$

0;0	$1^2;0$	3;0	0000	+	$\{1;2\ 3;0\ 0;1^2\}$	$2\sqrt{3}/n\sqrt{(n+2)(n+1)(n-1)}$
0;1	1;0	2;0	0000	+	+1	$\sqrt{2}/n\sqrt{(n+2)(n-1)}$
0;2	1;1	1;0	0000	+	$-\{1;2\ 2;0\ 0;1\}\{0;3\ 2;0\ 2;0\}$	$2/n\sqrt{(n+2)(n+1)(n-1)}$
0;21	0;1	1;1	0000	+	$\{0;2\ 1;0\ 1;0\}$	$\sqrt{6}/n(n-1)\sqrt{(n+2)(n+1)}$

$1;2\ \ 3;0\ \ 0;2$

0;0	2;0	3;0	0000	+	$\{1;2\ 3;0\ 0;2\}$	$2\sqrt{3}/(n+1)n\sqrt{n+2}$
0;1	1;0	2;0	0000	+	+1	$\sqrt{2(n+3)}/(n+1)n\sqrt{n+2}$
0;2	1;1	1;0	0000	+	$\{1;2\ 2;0\ 0;1\}\{0;3\ 2;0\ 1;0\}$	$2\sqrt{n+3}/(n+2)(n+1)\sqrt{n(n-1)}$
0;21	0;1	1;1	0000	+	$\{0;1\}$	$\sqrt{2(n+3)}/(n+1)n(n-1)\sqrt{n-2}$
0;3	0;1	1;1	0000	+	$\{0;1\}$	$2\sqrt{2}/(n+1)n\sqrt{n(n-1)}$
1;2	1;0	2;0	0000	+	$-\{0;1\}\{0;2\ 1;0\ 1;0\}$	$4/(n+2)(n+1)n(n-1)$

$1;2\ \ 1;1^2\ \ 1^2;0$

0;0	$0;1^2$	$1;1^2$	0000	+	$\{1;2\ 1;1^2\ 1^2;0\}$	$2/n\sqrt{(n+1)(n-1)(n-2)}$
0;1	0;21	$0;1^2$	0000	+	$\{0;1\}\{1;2\ 1;1\ 1;0\}\{1;2\ 2;0\ 0;1\}$	$\sqrt{6}/(n+1)n\sqrt{(n-1)(n-2)}$
1;0	0;1	1;1	0000	+	+1	$1/n\sqrt{(n+1)(n-1)}$
1;0	$1;1^2$	1;1	0000	+	$-\{0;1\}$	$\sqrt{2}/(n+1)(n-2)\sqrt{n(n-1)}$

$1;2\ \ 1;1^2\ \ 1^2;0$							
$1^2;0$	$1;1$	$1;0$	0000	+	$-\{1;2\ 1;1\ 1;0\}\{1;1^2\ 1^2;0\ 0;1\}\{1;1\ 1;0\ 0;1\}$	$\sqrt{2}/(n+1)\sqrt{n(n-1)(n-2)}$	
$1;1$	$0;2$	$0;1$	0000	+	$\{1;2\ 1;1\ 1;0\}\{1;1^2\ 1;1\ 1;0\}$	$\sqrt{2}/(n+1)n\sqrt{n-1}$	
$1^3;0$	$1;0$	$2;0$	0000	+	$\{0;1^2\ 1;0\ 1;0\}$	$2\sqrt{2}/n(n-1)\sqrt{(n+1)(n-2)}$	
$21;0$	$1;0$	$2;0$	0000	+	$\{0;2\ 1;0\ 1;0\}$	$\sqrt{2}/(n+1)n\sqrt{(n+2)(n-1)}$	
$1^2;1$	$0;1$	$1;1$	0r00	$(-)^r$	$-\chi_0^{1;1^2}\{0;1\}$	$\sqrt{2}(n-2)/(n+1)\sqrt{(n-1)n(n^3-2n^2-2n+2)}$	for r=0
					$-\chi_1^{1;1^2}\{0;2\ 1;0\ 1;0\}$	$2\sqrt{n-3}/(n+1)\sqrt{(n+2)n(n-1)(n-2)(n^3-2n^2-2n-2)}$	for r=1
$1;2\ \ 1;1^2\ \ 2;0$							
$0;0$	$0;2$	$1;1^2$	0000	+	$\{1;2\ 1;1^2\ 2;0\}$	$2/(n+1)n\sqrt{n-2}$	
$0;1$	$0;21$	$0;1^2$	0000	+	$\{0;1^2\ 1;0\ 1;0\}$	$\sqrt{2}/n(n-1)\sqrt{(n+1)(n-2)}$	
$0;1$	$0;3$	$0;1^2$	0000	+	$\{0;2\ 1;0\ 1;0\}$	$2\sqrt{2}/(n+1)n\sqrt{(n+2)(n-1)}$	
$1;0$	$0;1$	$1;1$	0000	+	$+1$	$1/n\sqrt{(n+1)(n-1)}$	
$1^2;0$	$1;1$	$1;0$	0000	+	$\{1;2\ 1;1\ 1;0\}\{1;1^2\ 1^2;0\ 0;1\}\{1;1\ 1;0\ 0;1\}$	$\sqrt{2}/n(n-1)\sqrt{n+1}$	
$1;1$	$0;2$	$0;1$	0000	+	$-\{1;2\ 1;1\ 1;0\}\{1;1^2\ 1;1\ 1;0\}$	$\sqrt{2}/(n-1)\sqrt{(n+2)(n+1)n}$	
$21;0$	$1;0$	$2;0$	0000	+	$\{0;1\}$	$\sqrt{6}/n(n-1)\sqrt{(n+2)(n+1)}$	
$2;1$	$0;1$	$1;1$	0000	+	$-\{0;1\}$	$\sqrt{2}/(n+2)(n-1)\sqrt{(n+1)n}$	

$1;2\ \ 1^2;1\ \ 1;1$

0;0	1;1	$1^2;1$	0000	+	$\{1;2\ 1^2;1\ 1;1\}$	$\sqrt{2}/(n+1)\sqrt{n(n-1)(n-2)}$
0;1	0;1	1;1	0000	+	+1	$1/(n+1)(n-1)$
0;1	$1;1^2$	1;1	0000	+	$-\{0;1\}\{0;2\ 1;0\ 1;0\}$	$\sqrt{2}/(n+1)(n-2)\sqrt{n(n-1)}$
1;0	$1^2;1$	$1^2;0$	0000	+	$-\{0;1\}$	$\sqrt{2}/(n+1)(n-2)\sqrt{n(n-1)}$
$0;1^2$	0;2	0;1	0000	+	$\{0;1\}\{1;2\ \ 2;0\ 0;1\}\{1;1^2\ 1^2;0\ 0;1\}$	$2/(n+1)(n-1)\sqrt{n}$
1;1	1;1	1;0	00r0	$(-)^r$	$\chi_r\{0;1\}\{1;2\ \ 2;0\ 0;1\}\{1;1^2\ 1^2;0\ 0;1\}$	$\frac{[n-2(1-\delta_r)]}{(n+1)(n-1)(n-2)}\sqrt{\frac{n-2\delta_r}{2n(n+2\delta_r)}}$
$1;1^2$	0;1	1;1	0r00	$(-)^r$	$\chi^{1;1^2}\{1;1\ 1;0\ 0;1\}\{0;2\ 1;0\ 1;0\}$ $-\chi_1^{1;1^2}$	$1/(n+1)(n-1)\sqrt{(n-2)(n^3-2n^2-2n+2)}$ for r=0 $\sqrt{2(n-3)}/n\sqrt{(n+2)(n-2)(n^3+2n^2-2n-2)}$ for r=1
$1^2;1$	1;0	2;0	0000	+	$\{0;1\}$	$\sqrt{2}/(n+1)n\sqrt{(n-1)(n-2)}$
1;2	0;1	1;1	0000	+	$-\{0;1\}\{0;1^2\ 1;0\ 1;0\}$	$\sqrt{2}/(n+2)(n-1)\sqrt{(n+1)n}$
2;1	1;0	2;0	0000	+	$-\{0;1\}$	$\sqrt{2}/(n+2)(n-1)\sqrt{(n+1)n}$

$1;2\ 1;2\ 1^2;0$

1;0	0;1	1;1	0000	+	+1	$1/(n-1)\sqrt{(n+2)n}$
$1^2;0$	0;0	2;1	0000	+	$\{1;2\ 1;2\ 1^2;0\}$	$2/n(n-1)\sqrt{n+2}$
1;1	0;2	0;1	0000	+	$\{0;1^2\ 1;0\ 1;0\}$	$\sqrt{2}/(n-1)\sqrt{(n+2)(n+1)n}$
21;0	1;0	2;0	0000	+	$-\{0;1\}\{0;2\ 1;0\ 1;0\}\{0;1^2\ 1;0\ 1;0\}$	$\sqrt{6}/n(n+1)\sqrt{(n+2)(n+1)}$
$1^2;1$	0;1	1;1	0000	+	$\{0;1\}$	$\sqrt{2}/n(n-1)\sqrt{(n+2)(n+1)}$
2;1	1;2	0;0	0000	+	$\{1;2\ 1;2\ 1^2;0\}$	$2/(n+2)n(n-1)$

1;2 1;2 2;0

1;0	0;1	1;1	0000	+	+1	$\sqrt{n+3}/(n+1)\sqrt{(n+2)n(n-1)}$
2;0	0;0	2;1	0000	+	{1;2 1;2 2;0}	$\sqrt{2}/(n-1)\sqrt{(n+2)(n+1)n}$
1;1	0;2	0;1	0000	+	{0;2 1;0 1;0}	$\sqrt{2(n+3)}/(n+2)(n+1)(n-1)$
21;0	1;0	2;0	0000	+	−{0;2 1;0 1;0}	$\sqrt{2(n+3)}/(n+1)n(n-1)\sqrt{n+2}$
3;0	1;0	2;0	0000	+	{0;2 1;0 1;0}	$2\sqrt{2}/(n+2)(n+1)\sqrt{n(n-1)}$
2;1	0;1	1;1	0r00	$(-)^r$	$\chi_0^{1;2}\{0;1\}$ $\chi_1^{1;2}$	$n\sqrt{2(n+3)}/(n+2(n+1)(n-1)\sqrt{n^3+2n^2-2n-2}$ for r=0 $2\sqrt{n-2}/(n+2)n(n-1)\sqrt{n^3+2n^2-2n-2}$ for r=1
2;1	1;2	0;0	0000	+	{1;2 1;2 2;0}	$2/(n+2)n(n-1)$

2;1 1;2 0;0

0;0	0;0	1;2	0000	+	{1;2}	$\sqrt{2/(n+2)n(n-1)}$
0;1	0;1	0;2	0000	+	{1;2 2;0 0;1}	$\sqrt{2}/n\sqrt{(n+2)(n-1)}$
1;0	1;0	1;1	0000	+	{1;2 1;1 1;0}	$\sqrt{2}/n\sqrt{(n+2)(n-1)}$
$0;1^2$	$0;1^2$	0;21	0000	+	{1;2 21;0 $0;1^2$}	$2/n(n-1)\sqrt{n+2}$
$0;1^2$	$0;1^2$	0;3	0000	+	{1;2 3;0 $0;1^2$}	$2/n(n-1)\sqrt{n+2}$
$1^2;0$	$1^2;0$	$1^2;1$	0000	+	{1;2 $1;1^2$ $1^2;0$}	$2/n(n-1)\sqrt{n+2}$
$1^2;0$	$1^2;0$	2;1	0000	+	{1;2 1;2 $1^2;0$}	$2/n(n-1)\sqrt{n+2}$
0;2	0;2	0;21	0000	+	{1;2 21;0 0;2}	$2/n\sqrt{(n+2)(n+1)(n-1)}$
0;2	0;2	0;3	0000	+	{1;2 3;0 0;2}	$2/n\sqrt{(n+2)(n+1)(n-1)}$
2;0	2;0	1;0	0000	+	{1;2 2;0 0;1}	$2/n\sqrt{(n+2)(n+1)(n-1)}$
2;0	2;0	$1^2;1$	0000	+	{1;2 $1;1^2$ 2;0}	$2/n\sqrt{(n+2)(n+1)(n-1)}$
2;0	2;0	2;1	0000	+	{1;2 1;2 2;0}	$2/n\sqrt{(n+2)(n+1)(n-1)}$
1;1	1;1	0;1	0000	+	{1;2 1;1 1;0}	$\sqrt{2}/(n-1)\sqrt{(n+2)(n+1)n}$
1;1	1;1	$1;1^2$	0000	+	{1;2 $1^2;1$ 1;1}	$\sqrt{2}/(n-1)\sqrt{(n+2)(n+1)n}$
1;1	1;1	1;2	rr00	$(-)^{r+r'}$	$\{2;1\ 1;2\ 1;1\ r\}\delta_{rr'}$	$\sqrt{2}/(n-1)\sqrt{(n+2)(n+1)n}$
21;0	21;0	$1^2;0$	0000	+	{1;2 21;0 $0;1^2$}	$\sqrt{6}/n(n-1)\sqrt{(n+2)(n+1)}$
21;0	21;0	2;0	0000	+	{1;2 21;0 0;2}	$\sqrt{6}/n(n-1)\sqrt{(n+2)(n+1)}$
3;0	3;0	$1^2;0$	0000	+	{1;2 3;0 $0;1^2$}	$2\sqrt{3}/(n+2)n\sqrt{(n+1)(n-1)}$

$2;1 \quad 1;2 \quad 0;0$

$3;0$	$3;0$	$2;0$	0000	+	$\{1;2\ 3;0\ 0;2\}$	$2\sqrt{3}/(n+2)n\sqrt{(n+1)(n-1)}$
$1;1^2$	$1;1^2$	$0;1^2$	0000	+	$\{1;2\ 1;1^2\ 1^2;0\}$	$2/n\sqrt{(n+2)(n+1)(n-1)(n-2)}$
$1;1^2$	$1;1^2$	$0;2$	0000	+	$\{1;2\ 1;1^2\ 2;0\}$	$2/n\sqrt{(n+2)(n+1)(n-1)(n-2)}$
$1^2;1$	$1^2;1$	$1;1$	0000	+	$\{1;2\ 1^2;1\ 1;1\}$	$2/n\ (n+2)(n+1)(n-1)(n-2)$
$1;2$	$1;2$	$0;1^2$	0000	+	$\{1;2\ 1;2\ 1^2;0\}$	$2/(n+2)n(n-1)$
$1;2$	$1;2$	$0;2$	0000	+	$\{1;2\ 1;2\ 2;0\}$	$2/(n+2)n(n-1)$
$2;1$	$2;1$	$0;0$	0000	+	$\{1;2\}$	$2/(n+2)n(n-1)$

$2;1 \quad 1;2 \quad 1;1$

$0;1$	$0;1$	$0;2$	000r	$(-)^r$	$\chi_0^{1;2}\{0;1\}\{0;1^2\ 1;0\ 1;0\}\{0;2\ 1;0\ 1;0\}$	$2(2n+1)/(n+1)n(n-1)\sqrt{(n+2)(n^3+2n^2-2n-2)}$	for r=0
					$\chi_1^{1;2}$	$\sqrt{2(n+3)(n-2)}/(n-1)\sqrt{(n+2)(n^3+2n^2-2n-2)}$	for r=1
$1;0$	$1;0$	$1;1$	000r	$(-)^r$	$\chi_0^{1;2}$	$\sqrt{n^3+2n^2-2n-2}/(n+1)n(n-1)\sqrt{n+2}$	for r=0
						0	for r=1
$2;0$	$2;0$	$1;0$	000r	$(-)^r$	$\chi_0^{1;2}\{0;1\}\{1;1\ 1;0\ 0;1\}$	$2\sqrt{n^3+2n^2-2n-2}/(n+2)(n+1)n(n-1)$	for r=0
						0	for r=1
$1;1$	$0;0$	$1;2$	0r0r'	$(-)^{r+r'}$	$\{2;1\ 1;2\ 1;1\ r\}\delta_{rr'}$	$\sqrt{2}/(n-1)\sqrt{(n+2)(n+1)n}$	
$1;1$	$1;1$	$0;1$	00rr'	$(-)^{r+r'}$	$\chi_r^{1;1}\chi_0^{1;2}\{1;1\ 1;1\ 1;1\ r\}$	$(n^2+2(1+\delta r)n+2)/(n+1)(n-1)$ $\times\sqrt{(n-2\delta r)/2(n+2)(n+2\delta r)n(n^3+2n^2-2n-2)}$	for r=0
					$\chi_r^{1;1}\chi_1^{1;2}\{1;1\ 1;1\ 1;1\ r\}$	$(n-2+2\delta r)/(n-1)$ $\times\sqrt{(n+3)(n-2\delta r)/(n+2)(n+2\delta r)n(n-2)(n^3+2n^2-2n-2)}$	for r=1

2;1 1;2 1;1

$1;1^2$	0;1	0;2	000r	$(-)^{r}$	$-\chi_0^{1;2}\{0;1\}$	$\sqrt{2(n+2)}/(n-1)\sqrt{(n+1)n(n^3+2n^2-2n-2)}$	for r=0
					$\chi_1^{1;2}\{0;1^2\ 1;0\ 1;0\}$	$2\sqrt{(n+3)}/(n-1)\sqrt{(n+2)(n+1)n(n-2)(n^3+2n^2-2n-2)}$	for r=1
$1^2;1$	1;0	1;1	000r	$(-)^{r}$	$\chi_0^{1;2}\{1;1\ 1;0\ 0;1\}\{0;2\ 1;0\ 1;0\}$	$1/(n+1)(n-1)\sqrt{(n+2)(n^3+2n^2-2n-2)}$	for r=0
					$-\chi_1^{1;2}$	$\sqrt{2(n+3)}/n\sqrt{(n+2)(n-2)(n^3+2n^2-2n-2)}$	for r=1
1;2	0;1	0;2	000r	$(-)^{r}$	$\chi_0^{1;2}\{0;1\}$	$n\sqrt{2(n+3)}/(n+2)(n+1)(n-1)\sqrt{n^3+2n^2-2n-2}$	for r=0
					$-\chi_1^{1;2}$	$2\sqrt{n-2}/(n+2)n(n-1)\sqrt{n^3+2n^2-2n-2}$	for r=1
2;1	1;0	1;1	0r0r'	$(-)^{r+r'}$	$-\chi_o^{1;2}\chi_o^{1;2}\{2;1\ 1;2\ 1;1\ 0\}$	$(n^2+6n+4)/(n+2)(n+1)(n-1)(n^3+2n^2-2n-2)$	for r=0 r'=0
					$\chi_o^{1;2}\chi_1^{1;2}\{2;1\ 1;2\ 1;1\ 0\}$	$(n+1)\ \ 2(n+3)(n-2)/(n+3)(n-1)(n^3+2n^2-2n-2)$	for r=0 r'=1
					$\chi_1^{1;2}\chi_1^{1;2}\{2;1\ 1;2\ 1;1\ 0\}\{0;1\}$	$2(2n+1)/(n+2)n(n-1)(n^3+2n^2-2n-2)$	for r=1 r'=1
2;1	2;1	0;0	00rr'	$(-)^{r+r'}$	$\{2;1\ 1;2\ 1;2\ r\}\delta_{rr'}$	$2/(n+2)n(n-1)$	

"I could put it in terms of three words:

illusion, delusion, and collusion."

Bohm 1982.

CHAPTER V

THE SYMMETRIC GROUP-UNITARY GROUP DUALITY THEORY

1. INTRODUCTION

The connection between the symmetric and unitary groups has been known since the work of Schur and Frobenius. Later, Weyl (1931, 1946) showed that the Young symmetrizers, developed for the symmetric groups, may be used to obtain the irreps of the unitary groups(see also Murnaghan 1938). Weyl discussed the concept of duality, giving numerous theorems concerned with irreps of both groups, and also, applications to the many-body system of f equivalent particles. Such systems arise in many areas from molecular physics to elementary particle physics.

However, the Schur function approach makes the duality more apparent. These functions (Schur 1901, Littlewood 1940) had been studied by Jacobi, Trudi, Kostka and others, under the name of bialternants, long before Schur showed their connection with the characters of the symmetric and unitary groups. The use of the purely combinatoric properties of Schur functions is still proving fruitful in obtaining new identities, and thus, new computational techniques for character theory (see King 1970, Wybourne 1970, Butler and King 1973a,b; King, Luan and Wybourne 1981, King and Wybourne 1982).

The duality goes further than that expressed by Schur functions. Many powerful equalities, between various transformation coefficients of the symmetric groups and the unitary groups, can be established. Jahn (1950) was the first of many nuclear shell model theorists to use the duality to compute the jm and j symbols of a unitary group, work which was much extended (Jahn 1954; Elliott, Hope and Jahn 1953; Kaplan 1962a,b; Horie 1964; Kramer and Seligman 1969b; Vanagas 1971). Their results are derived using the Young symmetrizers of the symmetric group as projectors for the unitary group. Kramer (1967) used explicit transformations between the bases defined in terms of different symmetric group chains to define his f symbol (our resubduction factor) for a symmetric group. He showed that the f symbols were, essentially, equivalent to recoupling coefficients (6j and 9j symbols) for any unitary group (Kramer 1968), and further, that f symbols were also equal to coupling coefficients (3jm symbols) for $U_{p+q} \supset U_p \times U_q$. The symmetry properties of the symmetric group f symbol, together with the duality result, gave the origin of the Regge symmetries for the 6j symbols of SU_2, and for the 3jm symbols of $SU_2 \supset U_1$ (equivalently $SO_3 \supset SO_2$) (Kramer and Seligman 1969a) A simpler formulation of the various transformations followed using the concept of double coset (DC) generators of the symmetric group (Kramer and Seligman 1969b). Sullivan (1973, 1975a,b, 1976, 1978 a,b, 1980) has formulated the general theory of DC decompositions, developing many more dualtiy results.

In this chapter, we further extend the Schur-Weyl duality. The group theory and transformation theory that we require for this has been given in Chapter II and III. Section 2 presents a construction of the Schur-Weyl basis using creation and annihilation operators. Three factorization lemmas are derived in Section 3. The symmetric group-unitary group duality factors, which arise in these lemmas, have been omitted or presumed to be unity by previous authors. The importance of the factors lies in the fact that they relate the phase and multiplicity freedoms within GH transformation theory of the symmetric groups to similar freedoms for the unitary groups. In Section 4, we give the complex conjugation and the transposition symmetries of the duality factors. The duality relations of Kramer and Seligman, and of Sullivan are derived directly from our lemmas in Section 5. The relations give extensions of the Regge symmetries of the SU_2 6j symbols and the $SU_2 \supset U_1$ 3jm symbols, to all unitary groups.

2. THE SCHUR-WEYL BASIS

The dual structures of the symmetric and unitary groups may be exhibited in the language of creation and annihilation operators (Jordan 1935; Schwinger 1952; Baird and Biedenharn 1963; Moshinsky 1963). One constructs a Hilbert space to carry representations of the symmetric group, S_f, and the unitary group, U_p. Lezuo (1972) has used such a realization to study $S_f \times U_3$. The creation operator formulation makes the Schur-Weyl duality quite apparent.

In this "second quantization" notation, the single particle basis states are given by boson (or fermion) creation operators acting on a suitably defined vacuum state $|0\rangle$

$$a_k^\dagger|0\rangle \quad (1 \leqslant k \leqslant p) \quad . \tag{V.2.1}$$

These operators have the usual commutation (anticommutation) relations

$$a_k \equiv (a_k^\dagger)^\dagger \ , \quad [a_k^\dagger, a_\ell^\dagger]_\pm = 0 = [a_k, a_\ell]_\pm \tag{V.2.2}$$

and

$$[a_k^\dagger, a_\ell]_\pm = \delta^k{}_\ell \tag{V.2.3}$$

Using these basic relations, we find the p^2 operators,

$$F^k{}_\ell \equiv a_k^\dagger a_\ell \qquad 1 \leqslant k, \ \ell \leqslant p \ , \tag{V.2.4}$$

satisfy (for either bosons or fermions) the commutation relations

$$[F^k{}_\ell, F^m{}_n] = \delta^m{}_\ell F^k{}_n - \delta^k{}_n F^m{}_\ell \quad . \tag{V.2.5}$$

Hence the operators, $F^k{}_\ell$, are closed under commutation and describe the Lie algebra of GL_p. However, we identify them as the generators of U_p, since we are interested in unitary transformations that preserve the orthogonality of the basis states. The p basis states, $a_k^\dagger|0\rangle$, transform as the defining irrep $\delta_p \equiv \{1\}$ of U_p and we may write

$$a_k^\dagger|0\rangle = |\delta_p k\rangle \qquad (k = 1\ldots p) \tag{V.2.6}$$

The f-particle basis states are constructed by a tensor product of f-boson (or fermion) creations operators acting on the vacuum state

$$|0\rangle \equiv |0\rangle \cdots |0\rangle \qquad \text{(f times)} \qquad (V.\ 2.7)$$

$$a^{\dagger 1}_{k_1} \ldots a^{\dagger f}_{k_f} \ |0\rangle \quad , \qquad (V.2.8)$$

where $a^{\dagger i}_{k_i}$ creates the i^{th} particle in the basis state $k_i (1 \leqslant i \leqslant f,\ 1 \leqslant k \leqslant p)$. These creation operators have similar properties to those of single particle creation operators

$$a^i_k \equiv (a^{\dagger i}_k)^\dagger \ , \quad [a^i_k,\ a^j_\ell]_\pm = 0 = [a^{\dagger i}_k,\ a^{\dagger j}_\ell]_\pm$$

and

$$[a^{\dagger i}_k,\ a^j_\ell]_\pm = \delta^j_i \quad \delta^k_\ell \quad . \qquad (V.2.9)$$

The p^f f-particle states transform according to the f-Kronecker product irrep $\delta^f_p \equiv \delta_p \times \ldots \times \delta_p$ (f times), of $U^f_p \equiv U_p \times \cdots \times U_p$ (f times). Thus, we may label the states as

$$a^{\dagger 1}_{k_1} \cdots a^{\dagger f}_{k_f} \ |0\rangle = |\delta_p k_1\rangle \cdots |\delta_p k_f\rangle$$

$$\equiv |\delta^f_p \ k_1 \ldots k_f\rangle \ 1 \leqslant k_1, \ldots, k_f \leqslant p \ . \qquad (V.2.10)$$

A realization of the generators of both U_p and S_f can be constructed from the creation and annihilation operators. The generators have well defined actions on all f-particle states. The set of p^2 operators

$$F^k{}_\ell \equiv \sum_{i=1}^{f} a^{\dagger i}_k \ a^i_\ell \qquad 1 \leqslant k, \leqslant \ p \qquad (V.\ 2.11)$$

generate, under commutation, the Lie algebra of U_p, while the transposition operators

$$\tau_{ij} = \sum_{k,\ell} a_k^{\dagger i} \; a_\ell^{\dagger j} \; a_\ell^i \; a_k^j \quad 1 \leqslant i,j \leqslant f \tag{V.2.12}$$

generate the symmetric group S_f. Thus, the f-Kronecker product space δ_p^f furnishes a p^f dimension representation space for both U_p and S_f. Most importantly, since each operator of U_p in this realization commutes with each operator of S_f, the space δ_p^f is a representation space for the direct product group $S_f \times U_p$, which we shall call the Schur-Weyl group. The standard result (Weyl 1931, Murnaghan 1938, Littlewood 1940) is that we have a unique decomposition of δ_p^f into subspaces, which transform irreducibly under the action of the operators of the Schur-Weyl group. Each irrep of $S_f \times U_p$ in δ_p^f can be labelled

$$\lambda(S_f) \times \lambda'(U_p) \tag{V.2.13}$$

where λ is a partition of f into not more than p parts. The result central to the duality is that each irrep, $\lambda(S_f)$ occurs with a unique irrep $\lambda'(U_p)$ and vice versa. The representation labels of symmetric and unitary groups are usually chosen so that this uniqueness is emphasised, that is, by using the same partition $\lambda' = \lambda$. The occurrence of each irrep $\lambda(S_f) \times \lambda(U_p)$ is multiplicity free. Hence we have the following transformation of basis for the space δ_p^f

$$|\delta_p^f \; k_1 \ldots k_f\rangle = \sum_{\lambda i \ell} |\delta_p^f \; \lambda i \lambda \ell\rangle\langle\delta_p^f \; \lambda i \lambda \ell|\delta_p^f \; k_1 \ldots k_f\rangle \tag{V.2.14}$$

where i (respectively ℓ) labels the basis of irrep space λ of S_f (respectively U_p). The explicit reduction may be obtained via the application of Young symmetrizers.

The action of the operators $\tau\times F$ of $S_f\times U_p$ on the basis, $\{|\delta_p^f\ \lambda i\lambda\ell>\}$, which we shall call the Schur-Weyl basis, is given by

$$\tau\times F|\delta_p^f\ \lambda i\lambda\ell> = |\delta_p^f\ \lambda i'\lambda\ell'>\ \lambda(\tau)_i^{i'}\ \lambda(F)_\ell^{\ell'} \qquad (V.2.15)$$

where $\lambda(\tau')_i^{i'}$ and $\lambda(F)_\ell^{\ell'}$ are elements of matrix irreps $\underset{\sim}{\lambda}(S_f)$ and $\underset{\sim}{\lambda}(U_p)$, respectively. For convenience, we denote the Schur-Weyl basis as

$$|\delta_p^f\ \lambda i\lambda\ell> \equiv |\delta_p^f\ \begin{matrix} i \\ \lambda \\ \ell \end{matrix}> \qquad (V.2.16)$$

No choice of basis within the irrep spaces of either S_f or U_p is implied in the above. Of course, special bases do exist for both groups. The most important are known as the Young-Yamanouchi basis, $S_f\supset S_{f-1}\times S_1\supset S_{f-2}S_1\times S_1\supset\ldots$, for the symmetric groups and the Gel'fand basis, $U_p\supset U_{p-1}\times U_1\supset U_{p-2}\times U_1\times U_1\supset\ldots$, for the unitary groups. The latter has been used extensively by both Moshinsky and Biedenharn and their collaborators. In the following, we obtain the results that are valid for bases chosen with respect to subgroups that are direct products of a less restricted nature.

3. DUALITY FACTORS FOR THREE GROUP-SUBGROUP CHAINS

We produce three types of transformations, which take the Schur-Weyl basis states, (2.16), into one of the following group-subgroup schemes:

(1) the dissociation of the space, δ_p^f into the direct product of $\delta_p^{f_1}$ with $\delta_p^{f_2}(f=f_1+f_2)$,

(2) the transformation $\delta_p^f \rightarrow \delta_{p_1}^f \times \delta_{p_2}^f$ $(p=p_1p_2)$ obtained by the reduction $\delta_p \rightarrow \delta_{p_1} \times \delta_{p_2}$ in $U_p \supset U_{p_1} \times U_{p_2}$,

(3) the transformation $\delta_p^f \rightarrow \bigoplus_t r_t\ \delta_q^t \times \delta_{\bar{q}}^{\bar{t}} (\bar{q}=p-q,\ \bar{t}=f-t)$ obtained by the reduction $\delta_p \rightarrow \delta_q \times 0_{\bar{q}} + 0_q \times \delta_{\bar{q}}$ in $U_p \supset U_q \times U_{\bar{q}}$. For q fixed, r_t distinguishes the $\frac{f!}{t!\bar{t}!}$ equivalent spaces $\delta_q^t \times \delta_{\bar{q}}^{\bar{t}}$.

The uniqueness of the Schur-Weyl basis determines three transformation factors, which we will call duality factors. The numerical values of these factors depend only on the phase and multiplicity within the GH transformation theory of the symmetric and unitary groups.

Consider the first group-subgroup scheme depicted in figure 3.1. The irrep space, δ_p^f, is isomorphic to the direct product space of $\delta_p^{f_1}$ and $\delta_p^{f_2}$ with $f=f_1+f_2$. Each Kronecker product group is reduced to its corresponding Schur-Weyl group. The subgroup, $S_{f_1} \times S_{f_2} \times U_p$ is obtained by the subduction $S_f \supset S_{f_1} \times S_{f_2}$ on the left side of figure 3.1, and by the coupling $U_p \times U_p \supset U_p$ on the right side. Both the coupling and subduction processes are given by the outer multiplication of Schur functions, (the Littlewood-Richardson rule, Littlewood 1940 p94).

$$\{\lambda\}.\{\mu\} = m_{\lambda\mu}^{\nu}\{\nu\} \quad . \tag{V.3.1}$$

It is well-known (Weyl 1931, Th 3, p339) that if the representation $\lambda \times \mu(U_p \times U_p)$ contains the irrep, $\mu(U_p)$, exactly

$m^{\nu}_{\lambda\mu}$ times, then, conversely, the irrep $\nu(S_f)$ contains

Figure V.3.1

$$\begin{tikzcd}
 & \delta_p^f(U_p^f) \arrow[dl] \arrow[rr, phantom, "\sim"] & & \delta_p^{f_1}\delta_p^{f_2}(U_p^{f_1}\times U_p^{f_2}) \arrow[dr] & \\
\nu\nu(S_f\times U_p) \arrow[drr, "b"'] & & & & \lambda\lambda\mu\mu(S_{f_1}\times U_p\times S_{f_2}\times U_p) \arrow[dll, "r"] \\
 & & \lambda\mu\nu(S_{f_1}\times S_{f_2}\times U_p) \arrow[d, "ijn"] & & \\
 & & & &
\end{tikzcd}$$

on subduction the irrep $\lambda\mu(S_{f_1}\times S_{f_2})$ exactly $m^{\nu}_{\lambda\mu}$ times. Comparing figure 3.1 to figure II.2.1, we find that the following lemma is just an application of (II.2.11)

Lemma I: The duality factor of figure 3.1 is given by

$$\left| \delta_p^{f_1}\delta_p^{f_2} \begin{matrix} ij \\ \lambda\mu \\ r\nu n \end{matrix} \right\rangle = \left| \delta_p^{f} \begin{matrix} b\lambda\mu ij \\ \nu \\ n \end{matrix} \right\rangle D_p(\lambda\mu,\nu)^{b}{}_{r} \qquad \text{(V.3.2)}$$

where we have written

$$D_p(\lambda\mu,\nu)^{b}{}_{r} \equiv \left\langle \delta_p^{f} \begin{matrix} b\lambda\mu \\ \nu \end{matrix} \right| \left. \delta_p^{f_1}\;\delta_p^{f_2} \begin{matrix} \lambda\mu \\ r\nu \end{matrix} \right\rangle \qquad \text{(V.3.3)}$$

The duality factor is an element of a square matrix, which depends on the partitions λ,μ,ν and the group ranks

f_1, f_2, f, and p. We do not show the symmetric group ranks as they are implicit in the partitions.

Our second duality factor is obtained by considering the transformation between the GH bases $U_p^f \supset S_f \times U_p \supset S_f \times U_{p_1} \times U_{p_2}$ and $U_p^f \supset U_{p_1}^f \times U_{p_2}^f \supset S_f \times U_{p_1} \times S_f \times U_{p_2} \supset S_f \times U_{p_1} \times U_p$ (see figure 3.2). The first basis involves the subduction of $\lambda(U_p)$ to $\lambda_1\lambda_2(U_{p_1} \times U_{p_2})$ with $p=p_1p_2$, and the second couples $\lambda_1\lambda_2(S_f \times S_f)$ to $\lambda(S_f)$. Both processes are given by the inner multiplication of Schur functions,

$$\{\lambda_1\} \circ \{\lambda_2\} = \sum_{\lambda} g^{\lambda}_{\lambda_1\lambda_2} \{\lambda\} \ . \tag{V.3.4}$$

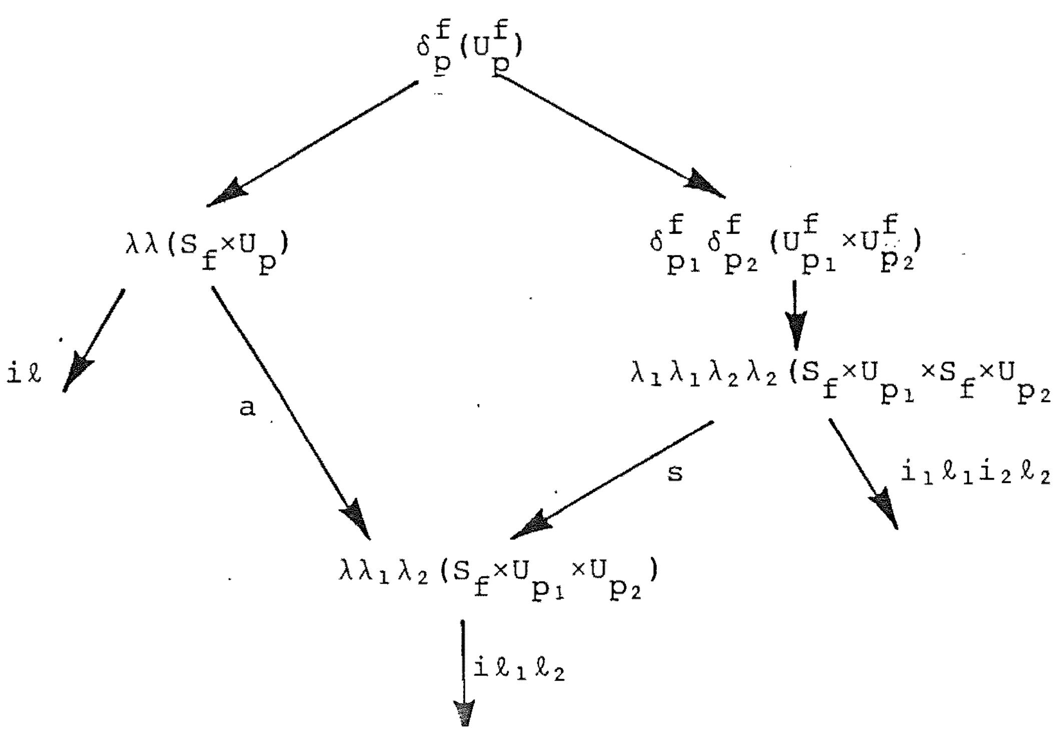
Lemma II : The duality factor of figure 3.2 is given by

$$\left| \delta^f_p \begin{matrix} i \\ \lambda \\ a\lambda_1\ell_1\lambda_2\ell_2 \end{matrix} \right\rangle = \left| \delta^f_{p_1} \delta^f_{p_2} \begin{matrix} s\lambda i \\ \lambda_1\lambda_2 \\ \ell_1\ell_2 \end{matrix} \right\rangle D_{p_1p_2}(\lambda, \lambda_1\lambda_2)^s{}_a \tag{V.3.5}$$

where we use the notation

$$D_{p_1p_2}(\lambda, \lambda_1\lambda_2)^s{}_a \equiv \left\langle \delta^f_{p_1} \delta^f_{p_2} \begin{matrix} s\ \lambda \\ \lambda_1\lambda_2 \end{matrix} \right| \left. \delta^f_p \begin{matrix} \lambda \\ a\lambda_1\lambda_2 \end{matrix} \right\rangle \tag{V.3.6}$$

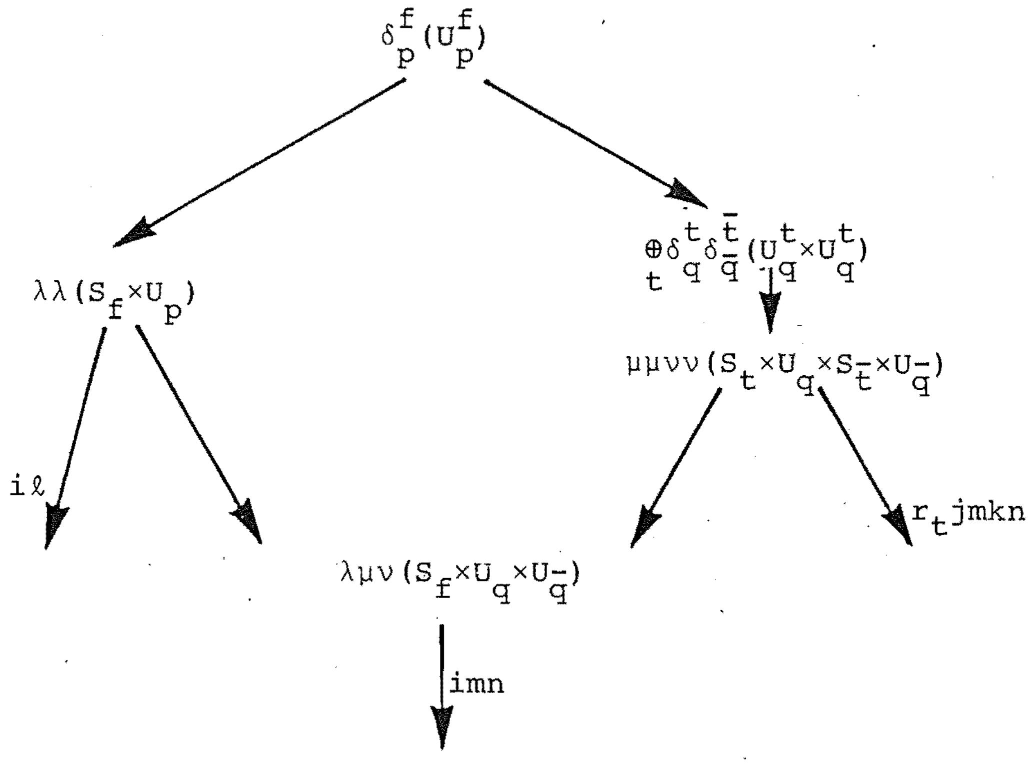
Figure V.3.2



This duality factor is also an element of a square matrix, which depends only on the partitions, $\lambda_1,\lambda_2,\lambda$, and the group ranks f,p_1p_2p. The rank of S_f is implicit in each $\lambda_1,\lambda_2,\lambda$, since each must be a partition of f.

The third duality factor is obtained from the decomposition $\delta_p \to \delta_q\cdot 0_{\bar{q}}+0_q\cdot\delta_{\bar{q}}$ under $U_p\supset U_q\times U_{\bar{q}}$, where $\bar{q}$=p-q and $0_q,0_{\bar{q}}$ are the identity irreps of U_q and $U_{\bar{q}}$, respectively. The f-Kronecker product space $\delta_p^f \to (\delta_q\cdot 0_{\bar{q}}+0_q\cdot\delta_{\bar{q}})^f$ is expanded as a direct sum of direct product spaces $\delta_q^t\times\delta_{\bar{q}}^{\bar{t}}$ ($\bar{t}$=f-t). That is, we have

Figure V.3.3



$(\delta_q\cdot 0_{\bar{q}}+0_q\delta_{\bar{q}})^f = \oplus\binom{f}{t}\delta_q^t\times\delta_{\bar{q}}^{\bar{t}}$ where for fixed t, the multiplicity of $\delta_q^t\times\delta_{\bar{q}}^{\bar{t}}$ in δ_p^f is $\binom{f}{t} = \dfrac{f!}{t!\bar{t}!}$. After reducing each Kronecker product group to its corresponding Schur-Weyl group (see figure 3.3), we can perform the induction $S_t S_{\bar{t}} \uparrow S_f$, for each t. This step can be understood by recognizing that the basis vectors, $|\delta^f_p \; r_t \; \delta^t_q \; \mu j \mu m \; \delta^{\bar{t}}_{\bar{q}} \; \nu k \nu n\rangle$ ($r_t = 1 \ldots \binom{f}{t}$) labels the multiplicity of $\delta^t_q \times \delta^{\bar{t}}_{\bar{q}}$ in δ^f_p), are, for varying ($r_t jk$) labels and fixed ($\mu m \nu n$) $U_q \times U_{\bar{q}}$ basis vector labels, the basis vectors of the induced representation $\mu\nu(S_t \times S_{\bar{t}} \uparrow S_f)$. (This replication of $S_t \times S_{\bar{t}}$ is the reason for placing r_t as a basis label of $S_t \times U_q \times S_{\bar{t}} \times U_{\bar{q}}$ in figure 3.3, rather than as a branching multiplicity label of $U^f_p \supset U^t_q \times U^{\bar{t}}_{\bar{q}}$). The induced representation space is reducible in S_f. Its constituent irreps are given by the outer multiplication of Schur functions, (3.1). The dimension of the induced space is (Wybourne 1970, eqn 45)

$$\frac{f!}{t!\bar{t}!} \; |\mu|_{S_t} \times \; |\nu|_{S_{\bar{t}}} = \sum_\lambda m^\lambda_{\mu\nu} |\lambda|_{S_f} \quad .$$

As explicit construction of the induced representation space can be obtained using coset theory (see Chapter II) but this is not necessary here.

An alternative labelling of basis vectors of δ^f_p is given by the GH basis $U^f_p \supset S_f \times U_p \supset S_f \times U_q \times U_{\bar{q}}$. The outer multiplication of Schur functions also determines the $U_p \supset U_q \times U_{\bar{q}}$ decomposition. The application of (II.2.11) to figure 3.3 provides the third lemma.

Lemma III : The duality factor of figure 3.3 is defined as

$$\left| \delta^f_p \; \begin{matrix} i \\ \lambda \\ a\mu m\nu n \end{matrix} \right\rangle = \left| \delta^f_p \delta^t_q \delta^{\bar{t}}_{\bar{q}} \; \begin{matrix} \uparrow s\lambda i \\ \mu\nu \\ mn \end{matrix} \right\rangle D_{q+\bar{q}}(\lambda,\mu\nu)^s{}_a$$

where $D_{q+\bar{q}}(\lambda,\mu\nu)^{s}{}_{a} = \left\langle \delta_p^f \delta_q^t \delta_{\bar{q}}^{\bar{t}} \; {}^{\dagger s\eta}_{\mu\nu} \middle| \delta_p^f \; {}^{\lambda}_{a\mu\nu} \right\rangle$

is an element of a square matrix depending on partition labels λ,μ,ν, and group ranks $f,q,\bar{q},p$.

We have defined three different duality factors, which can be seen to connect the multiplicity phase freedoms of the GH transformation theory of both the unitary and symmetric groups. Another way to interpret these lemmas is that they describe how to change any choice of unitary group multiplicity label into a symmetric group multiplicity label. The next section discusses the phase freedoms and symmetries of the duality factors.

4. PHASE FREEDOM AND SYMMETRIES OF THE DUALITY FACTORS.

In this section, we use the results of Chapter III to develop the phase freedom and symmetries of the duality factors defined in Section 3. In particular the complex conjugation symmetry requires the definition of a further three duality factors which arise from the relationship between the pure covariant/pure contravariant irrep discussed in Section IV.2.

In Section 3, we defined the duality factors from the f-Kronecker product δ_p^f which generates only the pure covariant irreps of U_p. We could have equally started with the pure contravariant (or complex conjugate) irrep of $\delta_p \equiv \{0;1\}$, that is $\bar{\delta}_p \equiv \{1;0\}$. The f-Kronecker product $\bar{\delta}_p^f$ generates the pure contravariant irreps of U_p, denoted

$\{\bar{\lambda}\} \equiv \{\lambda;0\}$. Although the irreps $\{\lambda\}$ and $\{\bar{\lambda}\}$ are inequivalent in U_p, we do not distinguish between the symmetric group label $[\lambda]$, denoting the symmetry type of the covariant irrep $\{\lambda\}\subset\delta_p^f$, and the symmetric group label denoting the symmetry type of the contravariant irrep $\{\bar{\lambda}\}\subset\bar{\delta}_p^f$. We denote the Schur-Weyl basis for pure contravariant irreps as

$$\left|\bar{\delta}_p^f \; \lambda i'\bar{\lambda}\ell'\right> \equiv \left|\bar{\delta}_p^f \; \begin{matrix} i' \\ \bar{\lambda} \\ \ell' \end{matrix}\right> \qquad \text{(V.4.1)}$$

A similar analysis, to that of Section 3, leads to following definitions of three more duality factors:

$$\text{I:} \quad \left|\bar{\delta}_p^{f_1}\delta_p^{f_2} \; \begin{matrix} i'j' \\ \bar{\mu}\;\bar{\nu} \\ r'\bar{\lambda}\ell \end{matrix}\right\rangle = \left|\bar{\delta}_p^f \; \begin{matrix} b'\mu i'\nu j' \\ \bar{\lambda} \\ \ell' \end{matrix}\right\rangle D_p(\bar{\mu}\bar{\lambda},\bar{\lambda})^{b'}{}_{r'},$$

$$\text{II:} \quad \left|\bar{\delta}_p^f \; \begin{matrix} i' \\ \bar{\lambda} \\ a'\bar{\lambda}_1\ell_1'\bar{\lambda}_2\ell_2' \end{matrix}\right\rangle = \left|\bar{\delta}_{p_1}^f\bar{\delta}_{p_2}^f \; \begin{matrix} s'\lambda i' \\ \bar{\lambda}_1\bar{\lambda}_2 \\ \ell_1'\ell_2' \end{matrix}\right\rangle D_{p_1p_2}(\bar{\lambda},\bar{\lambda}_1\bar{\lambda}_2)^{s'}{}_{a'},$$

$$\text{(V.4.2)}$$

$$\text{III:} \quad \left|\bar{\delta}_p^f \; \begin{matrix} i \\ \bar{\lambda} \\ a'\bar{\mu}m'\nu n' \end{matrix}\right\rangle = \left|\bar{\delta}_p^f\bar{\delta}_q^t\bar{\delta}_{\bar{q}}^{\bar{t}} \; \begin{matrix} \uparrow s'\lambda i' \\ \bar{\bar{\mu}}\bar{\bar{\nu}} \\ m'n' \end{matrix}\right\rangle D_{q+\bar{q}}(\bar{\lambda},\bar{\bar{\mu}}\bar{\bar{\nu}})^{s'}{}_{a'}$$

Each of these duality factors belongs to a square matrix of the same dimension as its pure covariant counterpart.

If we denote a different choice of duality factors by a circumflex, the phase freedom in the duality factors can be expressed by the following equivalence transformations:

(1) for pure covariant irreps

$$\text{(i)}\quad \hat{D}_p(\lambda\mu,\nu) = U[\nu,\lambda\mu]D_p(\lambda\mu,\lambda)U_p\{\lambda\mu,\nu\}^\dagger$$

$$\text{(ii)}\quad \hat{D}_{p_1p_2}(\lambda,\lambda_1\lambda_2) = U[\lambda_1\lambda_2,\lambda]D_{p_1p_2}(\lambda,\lambda_1\lambda_2)U_{p_1p_2}\{\lambda,\lambda_1\lambda_2\}^\dagger$$

$$\text{(iii)}\quad \hat{D}_{q+\bar{q}}(\lambda,\mu\nu) = U[\mu\nu\uparrow,\lambda]D_{q+\bar{q}}(\lambda,\mu\nu)U_{q+\bar{q}}\{\lambda,\mu\nu\}^\dagger \quad , \tag{V.4.3}$$

(2) for pure contravariant irreps

$$\text{(i)}\quad \hat{D}_p(\bar{\lambda}\bar{\mu},\bar{\nu}) = U[\nu,\lambda\mu]D_p(\bar{\lambda}\bar{\mu},\bar{\lambda})U_p\{\bar{\lambda}\bar{\mu},\bar{\nu}\}^\dagger$$

$$\text{(ii)}\quad \hat{D}_{p_1p_2}(\bar{\lambda},\bar{\lambda}_1\bar{\lambda}_2) = U[\lambda_1\lambda_2,\lambda]D_{p_1p_2}(\bar{\lambda},\bar{\lambda}_1\bar{\lambda}_2)U_{p_1p_2}\{\bar{\lambda},\bar{\lambda}_1\bar{\lambda}_2\}^\dagger$$

$$\text{(iii)}\quad \hat{D}_{q+\bar{q}}(\bar{\lambda},\bar{\mu}\bar{\nu}) = U[\mu\nu\uparrow,\lambda]D_{q+\bar{q}}(\bar{\lambda},\bar{\mu}\bar{\nu})U_{q+\bar{q}}\{\bar{\lambda},\bar{\mu}\bar{\nu}\}^\dagger \quad . \tag{V.4.4}$$

The unitary matrices U[...] and U{...} describe the phase freedoms for the symmetric and unitary group schemes respectively.

As a point of notation, we shall distinguish the transformation factors, such as the phase freedom transposition and complex conjugation factors, of the symmetric and unitary groups by surrounding the irrep labels by square and curly brackets respectively. This is in analogy to the common irrep notation of $[\lambda]$ for symmetry group irreps and $\{\lambda\}_p$ for unitary group irreps. We also use $[\lambda]$ and $\{\lambda\}_p$ to denote the 1j phase of the symmetric and unitary groups respectively. As mentioned in Section IV.2, no confusion arises since the meaning can be taken from the context.

We note that each symmetric group freedom in (4.3-4) has associated with it an infinite number of unitary group phase freedoms with the same partition labels but different group ranks, p, p_1, p_2, q and $\bar{q}$. Fixing the ranks of a unitary group scheme, we have two unitary group phase freedoms, that are related by the contravariant/covariant symmetry, for every symmetric group phase freedom. A consequence of this many-to-one relationship is the lack of phase freedom in choosing the duality factors. We shall return to this point in detail in Chapter VI.

The complex conjugation symmetry of duality factors is developed through the relationship between δ_p and $\delta_p^* = \bar{\delta}_p$, and, in general, between irreps $\{\lambda\}$ and $\{\lambda\}^* = \{\bar{\lambda}\}$ (cf. Section IV.2). We use the bar notation as a reminder that the duality theory deals only with the pure covariant and pure contravariant irreps of U_p. The complex conjugation symmetry of the duality factors follows from the correspondences

$$\begin{array}{llcl}
\text{(i)} & \delta_p^{f_1} \times \delta_p^{f_2} \simeq \delta_p^f & \sim & \bar{\delta}_p^{f_1} \times \bar{\delta}_p^{f_2} \simeq \bar{\delta}_p^f \\
\text{(ii)} & \delta_{p_1p_2}^f \supset \delta_{p_1}^f \times \delta_{p_2}^f & \sim & \bar{\delta}_{p_1p_2}^f \supset \bar{\delta}_{p_1}^f \times \bar{\delta}_{p_2}^f \\
\text{(iii)} & \delta_{q+\bar{q}}^f \supset \oplus \binom{f}{t} \delta_q^t \times \delta_{\bar{q}}^{\bar{t}} & \sim & \bar{\delta}_{q+\bar{q}}^f \supset \oplus \binom{f}{t} \bar{\delta}_q^t \times \bar{\delta}_{\bar{q}}^{\bar{t}}
\end{array} \qquad \text{(V.4.5)}$$

and these lead to the more explicit identities between duality factors of the covariant irreps and those of contravariant irreps:

$$\begin{array}{ll}
\text{(i)} & D_p(\bar{\lambda}\bar{\mu},\bar{\nu}) = A[\nu,\lambda\mu] D_p(\lambda\mu,\nu)^* \; A_p\{\lambda\mu,\nu\}^{\dagger *} \\
\text{(ii)} & D_{p_1p_2}(\lambda,\lambda_1\lambda_2) = A[\lambda_1\lambda_2,\lambda] D_{p_1p_2}(\lambda,\lambda_1\lambda_2)^* \; A_{p_1p_2}\{\lambda,\lambda_1\lambda_2\}^{\dagger *} \\
\text{(iii)} & D_{q+\bar{q}}(\bar{\lambda},\bar{\mu}\bar{\nu}) = A[\mu\nu\dagger,\lambda] D_{q+\bar{q}}(\lambda,\mu\nu)^* A_{q+\bar{q}}\{\lambda,\mu\nu\}^{\dagger *} \; .
\end{array} \qquad \text{(V.4.6)}$$

The complex conjugation factors A[...] and A{...} are the complex conjugation factors of the symmetric and unitary groups respectively. From the involutary nature of the complex conjugation operator and the symmetry of the complex conjugation factor given by (III.3.7), we obtain from (4.6) the quasi-ambivalence condition (Butler 1975, eq. 8.10) for the product of three 1j phases:

$$\begin{aligned}
&\text{(i)} \quad \{\lambda\}_p\{\mu\}_p\{\nu\}_p^* = [\nu][\lambda]^*[\mu]^* = +1 \\
&\text{(ii)} \quad \{\lambda\}_p\{\lambda_1\}_{p_1}^*\{\lambda_2\}_{p_2}^* = [\lambda_1][\lambda_2][\lambda]^* = +1 \qquad \text{(V.4.7)} \\
&\text{(iii)} \quad \{\lambda\}_p\{\mu\}_q^*\{\nu\}_{\bar{q}}^* = [\mu][\nu][\lambda]^* = +1 \quad .
\end{aligned}$$

The final equality in each equation follows from the real, orthogonal nature of symmetric group irreps. Hence, from the complex conjugation symmetry of the duality factors, we achieve the quasiambivalence condition for all unitary groups. This condition affords simplifications to the complex conjugation factors (see Section III.2).

The transposition symmetry for the duality factors is derived from the correspondences

$$\begin{aligned}
&\text{(i)} \quad U_p^{f_1} \times U_p^{f_2} \simeq U_p^f \quad \sim \quad U_p^{f_2} \times U_p^{f_1} \simeq U_p^f \\
&\text{(ii)} \quad U_{p_1p_2}^f \supset U_{p_1}^f \times U_{p_2}^f \quad \sim \quad U_{p_2p_1}^f \supset U_{p_2}^f \times U_{p_1}^f \qquad \text{(V.4.8)} \\
&\text{(iii)} \quad U_{q+\bar{q}}^f \supset \bigoplus_t \left(\tfrac{f}{t}\right) U_q^t \times U_{\bar{q}}^{\bar{t}} \quad \sim \quad U_{\bar{q}+q}^f \supset \bigoplus_t \left(\tfrac{f}{\bar{t}}\right) U_{\bar{q}}^{\bar{t}} \times U_q^t \quad .
\end{aligned}$$

The relationship between the two sets of duality factors are dependent on the transposition for the symmetric group schemes, denoted by T[...], and the unitary group schemes, denoted by T{...}. These can be shown to be:

(1) for pure covariant irreps

$$\begin{aligned}
&\text{(i)} \quad D_p(\mu\lambda,\nu) = T[\nu,\lambda\mu]D_p(\lambda\mu,\nu)T_p\{\lambda\mu,\nu\}^\dagger \\
&\text{(ii)} \quad D_{p_2p_1}(\lambda,\lambda_2\lambda_1) = T[\lambda_1\lambda_2,\lambda]D_{p_1p_2}(\lambda,\lambda_1\lambda_2)T_{p_1p_2}\{\lambda,\lambda_1\lambda_2\}^\dagger \\
&\text{(iii)} \quad D_{\bar{q}+q}(\lambda,\mu\nu) = T[\mu\nu\uparrow,\lambda]D_{q+\bar{q}}(\lambda,\mu\nu)T_{q+\bar{q}}\{\lambda,\mu\nu\}^\dagger \ , \qquad \text{(V.4.9)}
\end{aligned}$$

(2) for pure contravariant irreps

$$\begin{aligned}
&\text{(i)} \quad D_p(\bar{\mu}\bar{\lambda},\bar{\nu}) = T[\nu,\lambda\mu]D_p(\bar{\lambda}\bar{\mu},\bar{\nu})T_p\{\bar{\lambda}\bar{\mu},\bar{\nu}\}^\dagger \\
&\text{(ii)} \quad D_{p_2p_1}(\bar{\lambda},\bar{\lambda}_2\bar{\lambda}_1) = T[\lambda_1\lambda_2,\lambda]D_{p_1p_2}(\bar{\lambda},\lambda_1\lambda_2)T_{p_1p_2}\{\lambda,\lambda_1\lambda_2\}^\dagger \\
&\text{(iii)} \quad D_{q+\bar{q}}(\bar{\lambda},\bar{\nu}\bar{\mu}) = T[\mu\nu\uparrow,\lambda]D_{q+\bar{q}}(\bar{\lambda},\bar{\mu}\bar{\nu})T_{q+\bar{q}}\{\bar{\lambda},\bar{\mu}\bar{\nu}\}^\dagger \quad .
\end{aligned}$$

(V.4.10)

An important point to note, and one that recurs often in the Schur-Weyl duality, is the fact that the above symmetry relations equate an infinite number of complex conjugation factors, or transposition factors, of different unitary groups to one corresponding complex conjugation factor, or transposition factor, of the symmetric group. The exact numerical relationships would be determined once the duality factors were known. We take up this problem in Chapter VI. In the next section, we develop the powerful equalities between various transformation factors of the symmetric groups and those of the unitary groups, as we have done for the phase freedom, complex conjugation and transposition factors.

5. $S_f \times U_p$ DUALITY RELATIONS

In this section, we derive five duality relations which give the precise relationship between unitary group transformation factors and corresponding symmetric group transformation factors. The importance of these duality results is illustrated in the numerical equivalence via duality factors of an infinite number of transformation factors of different unitary groups. (See Bickerstaff et al. 1982 §4). These relations include the extension of the Regge symmetries of SU_2 and $SU_2 \supset U_1$ to symmetries for all unitary groups. After each statement of the five duality relations, we give an outline of its derivation. A key to understanding the various processes involved in each relation may be obtained by identifying the duality factors that appear.

Relation I : (Kramer identity)

The resubduction factor of the symmetric group scheme of figure 5.1a equals, to within four duality factors, a recoupling factor of U_p (figure 5.1b).

$$\langle \lambda b_1 \mu (b_2 \rho\sigma)\tau \,|\, \lambda b_3 \rho\nu (b_4 \sigma\tau) \rangle_{f_1+f_2+f_3}$$

$$= D_p(\rho\sigma,\mu)^{b_2}{}_{r_2}\; D_p(\mu\tau,\lambda)^{b_1}{}_{r_1}$$

$$\times \; \langle (\rho\sigma) r_2 \mu, \tau, r_1 \lambda \,|\, \rho(\sigma\tau) r_4\; \nu, r_3 \lambda \rangle_p$$

$$\times \; D_p(\sigma\tau,\nu)^{\dagger r_4}{}_{b_4}\; D_p(\rho\nu,\lambda)^{\dagger r_3}{}_{b_3}$$

The proof of this equality is obtained by considering the overlap of the two basis vectors

$$\left| \delta_p^{f_1} \delta_p^{f_2} \delta_p^{f_3} \begin{matrix} ijk \\ \rho\sigma\tau \\ r_2\mu, r_1\lambda\ell \end{matrix} \right\rangle \quad \text{and} \quad \left| \delta_p^{f_1} \delta_p^{f_2} \delta_p^{f_3} \begin{matrix} ijk \\ \rho\sigma\tau \\ r_4\nu, r_3\lambda\ell \end{matrix} \right\rangle$$

This gives the unitary group recoupling factor (see II.2). Using lemma I four times, we can replace the unitary group couplings by the symmetric group subductions. The product

<u>Figure V.5.1a</u>

$$\begin{tikzcd}
& \lambda(S_f) \arrow[dl, "b_1"'] \arrow[dr, "b_3"] & \\
\mu\tau(S_{f_{12}} \times S_{f_3}) \arrow[dr, "b_2"'] & & \rho\nu(S_{f_1} \times S_{f_{23}}) \arrow[dl, "b_3"] \\
& \rho\sigma\tau(S_{f_1} \times S_{f_2} \times S_{f_3}) &
\end{tikzcd}$$

$(f=f_{12}+f_3=f_1+f_{23}=f_1+f_2+f_3)$

Figure V.5.1b

$$\begin{tikzcd}
 & \rho\sigma\tau(U_p \times U_p \times U_p) \arrow[dl, "r_2"'] \arrow[dr, "r_4"] & \\
\mu\tau(U_p \times U_p) \arrow[dr, "r_1"'] & & \rho\nu(U_p \times U_p) \arrow[dl, "r_3"] \\
 & \lambda(U_p) &
\end{tikzcd}$$

multiplicity labels r_1, r_2, r_3, r_4, are changed to symmetric group branching multiplicity labels b_1, b_2, b_3, b_4. The overlap between the resulting basis vectors,

$$\left| \delta_p^f \begin{matrix} b_1\mu(b_2\rho i\sigma j)\tau k \\ \lambda \\ \ell \end{matrix} \right\rangle \quad \text{and} \quad \left| \delta_p^f \begin{matrix} b_3\rho i\nu(b_4\sigma j\tau k \\ \lambda \\ \ell \end{matrix} \right\rangle$$

produces the $S_{f_1+f_2+f_3}$ resubduction factor (see II.2.16).

Relation II :

A recoupling factor of S_f (figure 5.2a) equals, to within certain duality factors, a resubduction factor of the unitary group scheme of figure 5.2b.

$$\begin{aligned}
&<(\lambda_1\lambda_2)s_{12}\lambda_{12},\lambda_3,s\lambda|\lambda_1(\lambda_2\lambda_3)s_{23}\lambda_{23},s'\lambda>_f \\
&= D_{p_{12}p_3}(\lambda,\lambda_{12}\lambda_3)^s{}_a \; D_{p_1p_2}(\lambda_{12},\lambda_1\lambda_2)^{s_{12}}{}_{a_{12}} \\
&\times <\lambda a\lambda_{12}(a_{12}\lambda_1\lambda_2)\lambda_3|\lambda a'\lambda_1\lambda_{23}(a_{23}\lambda_2\lambda_3)>_{p_1p_2p_3} \\
&\times D_{p_1p_{12}}(\lambda,\lambda_1\lambda_{23})^{\dagger a'}{}_{s'} \; D_{p_2p_3}(\lambda_{23},\lambda_2\lambda_3)^{\dagger a_{23}}{}_{s_{23}} \; .
\end{aligned}$$

The proof is obtained by considering the overlap between the basis vectors,

$$\left|\delta^{f}_{p}\;\begin{matrix} i \\ \lambda \\ a\lambda_{12}(a_{12}\lambda_1\ell_1\lambda_2)\lambda_3\ell_3 \end{matrix}\right\rangle \quad \text{and} \quad \left|\delta^{f}_{p}\;\begin{matrix} i \\ \lambda \\ a'\lambda_1\ell_1\lambda_{23}(a_{23}\lambda_2\ell_2\lambda_3\ell_3) \end{matrix}\right\rangle$$

This gives the unitary group resubduction of $U_{p_1p_2p_3}$ (see II.2.16). We can replace each unitary group branching label, a_{12},a_{23},a,a', by a symmetric group product group multiplicity label, s_{12},s_{13},s,s', by using lemma II. The S_f recoupling factor is formed from the overlap of the resulting basis vectors (see II.2.14).

$$\left|\delta^{f}_{p_1}\delta^{f}_{p_2}\delta^{f}_{p_3}\;\begin{matrix} s_{12}\lambda_{12}s\lambda i \\ \lambda_1\lambda_2\lambda_3 \\ \ell_1\ell_2\ell_3 \end{matrix}\right\rangle \quad \text{and} \quad \left|\delta^{f}_{p_1}\delta^{f}_{p_2}\delta^{f}_{p_3}\;\begin{matrix} s_{23}\lambda_{23}s'\lambda i \\ \lambda_1\lambda_2\lambda_3 \\ \ell_1\ell_2\ell_3 \end{matrix}\right\rangle$$

Figure V.5.2a

$$\begin{tikzcd}
 & \lambda_1\lambda_2\lambda_3(S_f\times S_f\times S_f) \arrow[dl, "s_{12}"'] \arrow[dr, "s_{23}"] & \\
\lambda_{12}\lambda_3(S_f\times S_f) \arrow[dr, "s"'] & & \lambda_1\lambda_{23}(S_f\times S_f) \arrow[dl, "s'"] \\
 & \lambda(S_f) &
\end{tikzcd}$$

Figure V.5.2b

$$\begin{tikzcd}
 & \lambda(U_p) \arrow[dl, "a"'] \arrow[dr, "a'"] & \\
\lambda_{12}\lambda_3(U_{p_{12}}\times U_{p_3}) \arrow[dr, "a_{12}"'] & & \lambda_1\lambda_{23}(U_{p_1}\times U_{p_{23}}) \arrow[dl, "a_{23}"] \\
 & \lambda_1\lambda_2\lambda_3(U_{p_1}\times U_{p_2}\times U_{p_3}) &
\end{tikzcd}$$

$(p=p_{12}p_3=p_1p_{23}=p_1p_2p_3)$

Relation III :

A reinduction factor of the symmetric group scheme of figure 5.3a equals, to within certain duality factors, a resubduction factor of the unitary group scheme of figure 5.3b.

$$\langle (\rho\sigma)\uparrow s_1\mu,\tau,\uparrow s_2\lambda \,|\, \rho(\sigma\tau)\uparrow s_3\nu,\uparrow s_4\lambda\rangle_{f_1+f_2+f_3}$$

$$= D_{p_{12}+p_3}(\lambda,\mu\tau)^{s_2}{}_{a_2}\; D_{p_1+p_2}(\mu,\rho\sigma)^{s_1}{}_{a_1}$$

$$\times\; \langle \lambda a_2\mu(a_1\rho\sigma)\tau \,|\, \lambda a_3\rho\nu(a_4\sigma\tau)\rangle_{p_1+p_2+p_3}$$

$$\times\; D_{p_1+p_{23}}(\lambda,\rho\sigma)^{\dagger a_4}{}_{s_4}\; D_{p_2+p_3}(\nu,\sigma\tau)^{\dagger a_3}{}_{s_3}$$

<u>Figure V.5.3a</u>

$$\begin{tikzcd}
 & \rho\sigma\tau(S_{f_1}\times S_{f_2}\times S_{f_3})\uparrow S_f \arrow[dl, "s_1"'] \arrow[dr, "s_3"] & \\
\mu\tau(S_{f_{12}}\times S_{f_3})\uparrow S_f \arrow[dr, "s_2"'] & & \rho\nu(S_{f_1}\times S_{f_{23}})\uparrow S_f \arrow[dl, "s_4"] \\
 & \lambda(S_f) &
\end{tikzcd}$$

$(f=f_{12}+f_3=f_1+f_{23}=f_1+f_2+f_3)$

<u>Figure V.5.3b</u>

$$\begin{tikzcd}
 & \lambda(U_p) \arrow[dl, "a_2"'] \arrow[dr, "a_4"] & \\
\mu\tau(U_{p_{12}}\times U_{p_3}) \arrow[dr, "a_1"'] & & \rho\nu(U_{p_1}\times U_{p_{23}}) \arrow[dl, "a_3"] \\
 & \rho\sigma\tau(U_{p_1}\times U_{p_2}\times U_{p_3}) &
\end{tikzcd}$$

$(p=p_{12}+p_3=p_1+p_{23}=p_1+p_2+p_3)$

The proof is initiated with the overlap of the basis vectors

$$\left| \delta^f_p \begin{matrix} i \\ \lambda \\ a_2\mu(a_1\rho\ell\sigma m)\tau n \end{matrix} \right\rangle \quad \text{and} \quad \left| \delta^f_p \begin{matrix} i \\ \lambda \\ a_4\rho\ell\nu(a_3\sigma m\tau n) \end{matrix} \right\rangle .$$

This defines the resubduction factor of the unitary group, $U_{p_1+p_2+p_3}$ (see II.2.16). The unitary group branching multiplicity labels, a_1,a_2,a_3,a_4, are replaced by branching multiplicity labels, s_1,s_2,s_3,s_4, respectively, which are associated with symmetric group induction as determined by lemma III. The overlap of the two resulting basis vectors,

$$\left| \delta^f_p \delta^{f_1}_{p_1} \delta^{f_2}_{p_2} \delta^{f_3}_{p_3} \begin{matrix} \uparrow s_1\mu,\uparrow s_2\lambda i \\ \rho\sigma\tau \\ \ell mn \end{matrix} \right\rangle \quad \text{and} \quad \left| \delta^f_p \delta^{f_1}_{p_1} \delta^{f_2}_{p_2} \delta^{f_3}_{p_3} \begin{matrix} \uparrow s_3\nu,\uparrow s_4\lambda i \\ \rho\sigma\tau \\ \ell mn \end{matrix} \right\rangle$$

gives the reinduction factor of the symmetric group $(S_{f_1}\times S_{f_2}\times S_{f_3})\uparrow S_f$ (see II.5.2).

Relation IV :

A coupling factor (3jm symbol) of $S_f\supset S_{f_1}\times S_{f_2}$ (figure 5.4a) equals, to within various duality factors, a coupling factor (3jm symbol) of $U_p\supset U_{p_1}\times U_{p_2}$ (figure 5.4b).

$$\left\langle \begin{matrix} \lambda_1\lambda_2 \\ b_1b_2 \\ \mu_1\nu_1\mu_2\nu_2 \\ s_1s_2 \\ \mu\nu \end{matrix} \middle| \begin{matrix} \lambda_1\lambda_2 \\ s \\ \lambda \\ b \\ \mu\nu \end{matrix} \right\rangle^{f}_{f_1+f_2}$$

$$= D_{p_1p_2}(\mu,\mu_1\mu_2)^{s_1}{}_{a_1} D_{p_1p_2}(\nu,\nu_1\nu_2)^{s_2}{}_{a_2} D_{p_1}(\mu_1\nu_1,\lambda_1)^{b_1}{}_{r_1} D_{p_2}(\mu_2\nu_2,\lambda_2)^{b_2}{}_{r_2}$$

$$\times \left\langle \begin{array}{c} \mu\nu \\ a_1 a_2 \\ \mu_1\mu_2\nu_1\nu_2 \\ r_1 r_2 \\ \lambda_1\lambda_2 \end{array} \middle| \begin{array}{c} \mu\nu \\ r \\ \lambda \\ a \\ \lambda_1\lambda_2 \end{array} \right\rangle^{p}_{p_1 p_2}$$

$$D_p(\mu\nu,\lambda)^{\dagger r}{}_{b} \; D_{p_1p_2}(\lambda_1\lambda_2,\lambda)^{\dagger a}{}_{s}$$

Figure V.5.4a

$$\begin{tikzcd}
 & \lambda_1\lambda_2(S_f \times S_f) \arrow[dl, "b_1 b_2"'] \arrow[dr, "s"] & \\
\mu_1\nu_1\mu_2\nu_2(S_{f_1}\times S_{f_2}\times S_{f_1}\times S_{f_2}) \arrow[dr, "s_1 s_2"'] & & \lambda(S_f) \arrow[dl, "b"] \\
 & \mu\nu(S_{f_1}\times S_{f_2}) &
\end{tikzcd}$$

$(f=f_1+f_2)$

Figure V.5.4b

$$\begin{tikzcd}
 & \mu\nu(U_p \times U_p) \arrow[dl, "a_1 a_2"'] \arrow[dr, "r"] & \\
\mu_1\mu_2\nu_1\nu_2(U_{p_1}\times U_{p_2}\times U_{p_1}\times U_{p_2}) \arrow[dr, "r_1 r_2"'] & & \lambda(U_p) \arrow[dl, "a"] \\
 & \lambda_1\lambda_2(U_{p_1} U_{p_2}) &
\end{tikzcd}$$

$(p=p_1p_2)$

Consider the overlap between the two basis vectors

$$\left|\delta_p^{f_1}\delta_p^{f_2}\begin{array}{l} jk \\ \mu\nu \\ r\lambda a\lambda_1\ell_1\lambda_2\ell_2\end{array}\right\rangle \quad\text{and}\quad \left|\delta_p^{f_1}\delta_p^{f_2}\begin{array}{l} jk \\ \mu\nu \\ a_1a_2\mu_1\mu_2\nu_1\nu_2r_1r_2\lambda_1\ell_1\lambda_2\ell_2\end{array}\right\rangle .$$

This gives the $U_p\supset U_{p_1}\times U_{p_2}$ coupling factor. To each of the basis vectors, we apply both lemma I and lemma II. Each product multiplicity labels of the unitary group becomes a symmetric group branching multiplicity label and vice versa. We then form the scalar product of the resulting basis vectors,

$$\left|\delta_{p_1}^{f}\delta_{p_2}^{f}\begin{array}{l} s\lambda b\mu j\nu k \\ \lambda_1\lambda_2 \\ \ell_1\ell_2\end{array}\right\rangle \quad\text{and}\quad \left|\delta_{p_1}^{f}\delta_{p_2}^{f}\begin{array}{l} b_1b_2\mu_1\nu_1\mu_2\nu_2s_1s_2\mu j\nu k \\ \lambda_1\lambda_2 \\ \ell_1\ell_2\end{array}\right\rangle ,$$

which gives the $S_f\supset S_{f_1}\times S_{f_2}$ coupling factor.

Relation V :

An induction factor of $S_{f_1}\times S_{f_2}\uparrow S_f$ (figure 5.5a) equals, to within certain duality factors, a coupling factor of $U_p\supset U_q\times U_{\bar{q}}$ (figure 5.5b).

$$\left\langle\begin{array}{c} \mu\nu\uparrow \\ b_1b_2 \\ \sigma\tau\omega\zeta\uparrow \\ s_1s_2 \\ \rho\upsilon\end{array}\right|\left.\begin{array}{c} \mu\nu\uparrow \\ s \\ \lambda \\ b \\ \rho\upsilon\end{array}\right\rangle_{t+\bar{t}}^{f_1+f_2}$$

$$=D_{q+\bar{q}}(\rho,\sigma\tau)^{s_1}{}_{a_1}D_{q+\bar{q}}(\upsilon,\omega\zeta)^{s_2}{}_{a_2}D_q(\sigma\omega,\mu)^{b_1}{}_{r_1}D_{\bar{q}}(\tau\zeta,\nu)^{b_2}{}_{r_2}$$

$$\left\langle \begin{array}{c|c} \rho\upsilon & \rho\upsilon \\ a_1a_2 & r \\ \sigma\tau\omega\zeta & \lambda \\ r_1r_2 & a \\ \mu\nu & \mu\nu \end{array} \right\rangle_{q+\bar{q}}$$

$$D_{q+\bar{q}}(\rho\upsilon,\lambda)^{\dagger r}{}_{b} \quad D_{q+\bar{q}}(\lambda,\mu\nu)^{\dagger a}{}_{s}$$

Figure V.5.5a

$$\begin{tikzcd}[column sep=small] & \mu\nu(S_{f_1}\times S_{f_2})\uparrow S_f \arrow[dl, "b_1b_2"'] \arrow[dr, "s"] & \\ \sigma\tau\omega\zeta(S_{t_1}\times S_{\bar{t}_1}\times S_{t_2}\times S_{\bar{t}_2})\uparrow S_t\times S_{\bar{t}} \arrow[dr, "s_1s_2"'] & & \lambda(S_f) \arrow[dl, "b"] \\ & \rho\upsilon(S_t\times S_{\bar{t}}) & \end{tikzcd}$$

$(f=f_1+f_2=t_1+\bar{t}_1+t_2+\bar{t}_2=t+\bar{t})$

Figure V.5.5b

$$\begin{tikzcd} & \rho\upsilon(U_p\times U_p) \arrow[dl, "a_1a_2"'] \arrow[dr, "r"] & \\ \sigma\tau\omega\zeta(U_q\times U_{\bar{q}}\times U_q\times U_{\bar{q}}) \arrow[dr, "r_1r_2"'] & & \lambda(U_p) \arrow[dl, "a"] \\ & \mu\nu(U_q\times U_{\bar{q}}) & \end{tikzcd}$$

$(p=q+\bar{q})$

The proof follows by taking the scalar product of the basis vectors

$$\left|\begin{array}{ll} & ij \\ \delta_p^{f_1}\delta_p^{f_2} & \rho\upsilon \\ & (a_1\sigma\tau;a_2\omega\zeta)r_1r_2\mu m\nu n \end{array}\right\rangle \quad\text{and}\quad \left|\begin{array}{ll} & ij \\ \delta_p^{f_1}\delta_p^{f_2} & \rho\upsilon \\ & r\lambda a\mu m\nu n \end{array}\right\rangle ,$$

which defines the $U_p \supset U_q U_{\bar{q}}$ coupling factor. Applying lemmas I and III to change multiplicity labels of the unitary groups to those of the symmetric group, we obtain the resulting basis vectors

$$\left|\begin{array}{ll} & \uparrow b_1b_2\sigma\tau\omega\zeta\uparrow s_1s_2\rho i\upsilon j \\ \delta_t^f\delta_q^t\delta_{\bar{q}}^{\bar{t}} & \mu\nu \\ & mn \end{array}\right\rangle \quad\text{and}\quad \left|\begin{array}{ll} & \uparrow s\lambda b\rho i\upsilon j \\ \delta_p^f\delta_q^t\delta_{\bar{q}}^{\bar{t}} & \mu\nu \\ & mn \end{array}\right\rangle .$$

The scalar product of these vectors gives the induction factor for the symmetric group scheme of figure 5.5a (see II.6.18).

Thus, we have shown the precise relationship between various symmetric group transformation factors and combinatorically equivalent unitary group transformation factors. We summarise these in table 5.1. The identification of the unitary and symmetric group transformation factors of relation I was first demonstrated by

Table V.5.1 Duality Relationships

I :	resubduction $S_{f_1+f_2+f_3}$	$\simeq$ recoupling (6j) U_p
II :	recoupling (6j) S_f	$\simeq$ resubduction $U_{p_1p_2p_3}$
III :	reinduction $S_{f_1}\times S_{f_2}\times S_{f_3}\uparrow S_f$	$\simeq$ resubduction $U_{p_1+p_2+p_3}$
IV :	coupling (3jm) $S_{f_1+f_2}$	$\simeq$ coupling (3jm) $U_{p_1p_2}$
V :	induction $S_{f_1}\times S_{f_2}\uparrow S_f$	$\simeq$ coupling (3jm) $U_{p_1+p_2}$

Kramer (1968 eq. ..6) for bipartition irreps of SU_2 and S_f. Sullivan (1973 eq. 4.4) formulated the identification for all irreps of SU_p and S_f. He has also given (Sullivan 1976 eq. 4.11) the identification of factors appearing in relation V in the most general form, although, for bipartition irreps, this relation underlies the derivation of the Regge symmetries of $U_2 \supset U_1 \times U_1$ by Kramer and Seligman (1969a eq. 3.8). Sullivan (1975a eq. 10) has identified the resubduction factor (his DCME) of $U_{p_1+p_2+p_3}$ with the resubduction (his DCME) of $S_{f_1+f_2+f_3}$. In relation III, we have the appearance of the reinduction factor of $S_{f_1} \times S_{f_2} \times S_{f_3}$. The relationship between these two symmetric group transformation factors has been discussed in Section II.5. Relation IV, which identifies coupling factors of $S_{f_1+f_2}$ and $U_{p_1 p_2}$ has been given by Chen (1981 eq. 31).

The presence of the three duality factors complicates the dual relationship. In the next chapter, we study these factors in more detail and show that in connection with the $U_n \simeq U_1 \times SU_n$ isomorphism there is not sufficient freedom to choose all these factors.

CHAPTER VI

THE SYMMETRIES OF THE DUALITY FACTORS

1. INTRODUCTION

In this chapter, we discuss the possible matrix choices that can be given to both the pure covariant and pure contravariant duality factors. The choices clearly depend on the nature and number of the phase freedom restrictions imposed by choices of other transformation factors. The order in which one makes these matrix choices is also important in that different orders may give different matrix choices for the same transformation factors. For example, if we assume the choice given in Chapter III for the complex conjugation and transposition factors of the unitary group and symmetric group schemes, the duality factors must be determined subject to the phase freedom restrictions imposed by these choices and in conjunction with the complex conjugation and transposition symmetries given by equations (V.4.6) and (V.4.9-10). Alternatively, we can assume total phase freedom and first choose all the duality factors unity and second determine the complex conjugation and transposition factors of both unitary group and symmetric group schemes subject to the restrictions and symmetries imposed by these choices of duality factors. The matrix choices are not necessarily the same for the two procedures.

One important symmetry that we shall be considering in connection with the duality symmetry is the $U_p \simeq U_1 \times SU_p$ isomorphism. This has been discussed in Chapter IV with regard to the Racah-Wigner coupling algebra. In Section 2, we give a detailed a-count of this isomorphism defining the isomorphism factors for the coupling scheme and the two subduction schemes of the unitary groups. In Section 3 we define our notation and terminology which will be used in Section 4 to discuss the interplay of the duality and isomorphism symmetries. Two different procedures in determining matrix choices for the duality factors and the isomorphism factors are discussed. The important result here is that we show there is not enough freedom to choose all duality/isomorphism factors.

2. THE $U_p \simeq U_1 \times SU_p$ ISOMORPHISM

In this section, we discuss in more detail, the $U_p \simeq U_1 \times SU_p$ isomorphism given by (IV.2.3). In particular we define three transformation factors appropriate to the coupling scheme $U_p \times U_p \supset U_p$, and the two subduction schemes $U_{p_1 p_2} \supset U_{p_1} \times U_{p_2}$ and $U_{p+\bar{q}} \supset U_q \times U_{\bar{q}}$. These factors relate the phase freedom and symmetries of the unitary group schemes with the phase freedom and symmetries of the special unitary group schemes. For this section, we use the standard irrep labels for U_p.

The $U_p \simeq U_1 \times SU_p$ isomorphism allows us to write the basis vectors of the irrep space $\{\lambda\}$ of U_p as

$$|\lambda\ell\rangle = |f\lambda' 0k'\rangle\langle f\lambda' 0k'|\lambda\ell\rangle \qquad \text{(VI.2.1)}$$

where 0 labels the 1 dimensional irrep space for $\{f\}$ of U_1.

(The prime on any label denotes transformation properties under SU_p). The matrix invariance requirement forces the transformation coefficient appearing in (2.1) to be diagonal,

$$<f\lambda' ok'|\lambda\ell> = <f\lambda'|\lambda>\delta^k{}_\ell \tag{VI.2.2}$$

where $<f\lambda'|\lambda>$ is a phase which can always be chosen +1. Using (2.1), we can factorize the coupling and subduction schemes of the unitary group in accordance with the following relations

$$\begin{aligned}
&\text{(i)} \quad |\lambda\mu r\nu(f\nu'n')>=|f_1\lambda'f_2\mu'r'f\nu'n><f_1\lambda'f_2\mu'r'f\nu'|\lambda\mu r\nu(f\nu')> \\
&\text{(ii)} \quad |\lambda a\lambda_1\lambda_2(f\lambda_1'\ell_1'\lambda_2'\ell_2')>=|f\lambda'a'\lambda_1'\ell_1'\lambda_2'\ell_2'><f\lambda'a'\lambda_1'\lambda_2'|\lambda a\lambda_1\lambda_2(f\lambda_1'\lambda_2')> \\
&\text{(iii)} \quad |\lambda a\mu\nu(f\mu'm'\nu'n')>=|f\lambda'a'\mu'm'\nu'n'><f\lambda'a'\mu'\nu'|\lambda a\mu\nu(f\mu'\nu')>
\end{aligned} \tag{VI.2.3}$$

Note, $\{f\}$ is the resultant U_1 irrep after coupling all U_1 irreps originating from the isomorphism. The transformation factors appearing in (2.3) will be called isomorphism factors. For compactness, we write them in matrix form as

$$\text{(i)}\ F_p\{\lambda\mu,\times\}^{r'}{}_r,\ \text{(ii)}\ F_{p_1p_2}\{\lambda\ \lambda_1\lambda_2\}^{a'}{}_a,\ \text{(iii)}\ F_{q+\bar{q}}\{\lambda,\mu\nu\}^{a'}{}_a \tag{VI.2.4}$$

The SU_p irreps are obtained by applying (IV.2.2) on the U_p irrep labels, which form the arguments of the isomorphism factors. The symmetries of phase freedom, complex conjugation and transposition can be derived for these isomorphism factors, using similar arguments to those given in Chapter III. We state the following results without proof:

(1) the phase freedom

$$\text{(i)}\quad \hat{F}_p\{\lambda\mu,\nu\}=U_1\{f_1f_2,f\}U_p'\{\lambda'\mu',\nu'\}F_p\{\lambda\mu,\nu\}U_p\{\lambda\mu,\nu\}^\dagger$$

$$\text{(ii)}\quad \hat{F}_{p_1p_2}\{\lambda,\lambda_1\lambda_2\}=U'_{p_1p_2}\{\lambda',\lambda_1'\lambda_2'\}F_{p_1p_2}\{\lambda,\lambda_1\lambda_2\}U_{p_1p_2}\{\lambda,\lambda_1\lambda_2\}^\dagger$$

$$\text{(iii)}\quad \hat{F}_{q+\bar{q}}\{\lambda,\mu\nu\}=U'_{q+\bar{q}}\{\lambda',\mu'\nu'\}F_{q+\bar{q}}\{\lambda,\mu\nu\}U_{q+\bar{q}}\{\lambda,\mu\nu\}^\dagger$$

(VI.2.5)

(2) the complex conjugation symmetry

$$\text{(i)}\quad F_p\{\lambda^*\mu^*,\nu^*\}=A_1'\{f_1f_2,f\}A_p\{\lambda'\mu',\nu'\}F_p\{\lambda\mu,\nu\}^*A_p\{\lambda\mu,\nu\}^{\dagger *}$$

$$\text{(ii)}\quad F_{p_1p_2}\{\lambda^*,\lambda_1^*\lambda_2^*\}=A'_{p_1p_2}\{\lambda',\lambda_1'\lambda_2'\}F_{p_1p_2}\{\lambda,\lambda_1\lambda_2\}^*A_{p_1p_2}\{\lambda,\lambda_1\lambda_2\}^{\dagger *}$$

$$\text{(iii)}\quad F_{q+\bar{q}}\{\lambda^*,\mu^*\nu^*\}=A'_{q+\bar{q}}\{\lambda',\mu'\nu'\}F_{q+\bar{q}}\{\lambda,\mu\nu\}^*A_{q+\bar{q}}\{\lambda,\mu\nu\}^{\dagger *}$$

(VI.2.6)

and (3) the transposition symmetry

$$\text{(i)}\quad F_p\{\mu\lambda,\nu\}=T_1'\{f_1f_2,f\}T_p\{\lambda'\mu',\nu'\}F_p\{\lambda\mu,\lambda\}T_p\{\lambda\mu,\lambda\}^\dagger$$

$$\text{(ii)}\quad F_{p_2p_1}\{\lambda,\lambda_2\lambda_1\}=T'_{p_1p_2}\{\lambda',\lambda_1'\lambda_2'\}F_{p_1p_2}\{\lambda,\lambda_1\lambda_2\}T_{p_1p_2}\{\lambda,\lambda_1\lambda_2\}^\dagger$$

$$\text{(iii)}\quad F_{q+\bar{q}}\{\lambda,\nu\mu\}=T'_{q+\bar{q}}\{\lambda',\mu'\nu'\}F_{q+\bar{q}}\{\lambda,\mu\nu\}T_{q+\bar{q}}\{\lambda,\mu\nu\}^\dagger$$

(VI.2.7)

The special unitary group transformation factors are identified by the 'prime'. In the above we include isomorphism factors for which the arguments may be labelled by composite irreps. We discuss the matrix choices of the isomorphism factors along with those of the duality factors in Section 4.

3. SOME NOTATION AND TERMINOLOGY

In this section we introduce some notation and terminology to simplify our discussion of matrix choices for the duality factors and isomorphism factors.

The coupling, subduction and induction schemes of the unitary, special unitary and symmetric groups are given by outer S-function multiplication

$$\{\lambda\}\cdot\{\mu\} = \sum_{\nu} m^{\nu}_{\lambda\mu}\ \{\nu\} \tag{VI.3.1}$$

and inner S-function multiplication

$$\{\lambda_1\}\circ\{\lambda_2\} = \sum_{\lambda} g^{\lambda}_{\lambda_1\lambda_2}\{\lambda\} \tag{VI.3.2}$$

The decomposition is labelled by three partitions - $(\lambda),(\mu),(\nu)$ for outer multiplication and $(\lambda_1),(\lambda_2),(\lambda)$ for inner multiplication. The three partitions which describe this decomposition and which have non-zero multiplicity shall be a triple denoted $(\xi\zeta\eta)$. (Note, a triad refers only to coupling schemes and is obtained from a triple $(\xi\zeta\eta)$ as $(\xi\zeta\eta^*)$). For example, for the unitary group subduction $(\lambda,\lambda_1\lambda_2)$ is a triple if $g^{\lambda}_{\lambda_1\lambda_2} > 0$ and for the symmetric group induction $(\mu\nu\uparrow,\lambda)$ is a triple if $m^{\lambda}_{\mu\nu} > 0$. The comma is used to separate the group labels from the subgroup labels. Unless otherwise stated, we shall assume $(\xi\zeta\eta)$ is a unitary group triple labelled by pure covariant irrep labels. We shall pre-subscript an R to the triple to denote the rank dependence. Thus ${}_R(\xi\zeta\eta)$ implies

(1) for coupling $R = U_p\times U_p\supset U_p$ with $\xi\times\zeta\supset\eta$

(2) for subduction $R = U_{p_1p_2}\supset U_{p_1}\times U_{p_2}$ with $\xi\supset\zeta\eta$

(3) for subduction $R = U_{q+\bar{q}}\supset U_q\times U_{\bar{q}}$ with $\xi\supset\zeta\times\eta$.

Hence ${}_{R_1}(\xi\zeta\eta)$ and ${}_{R_2}(\xi\zeta\eta)$ denote triples with the same partition labels but different unitary group schemes.

The triples $(d\cdot\xi\zeta\eta)$ and ${}_R(\xi'\zeta'\eta')$ will denote respectively the duality-related symmetric group triple and the special unitary group triple obtained from the unitary group triple ${}_R(\xi\zeta\eta)$. The triple $(d\cdot\xi\zeta\eta)$ which has clearly no unitary group rank dependency will be

(1) $(d\cdot\xi\zeta\eta)=(\eta,\xi\zeta)$ if ${}_R(\xi\zeta\eta)$ is a coupling $U_p\times U_p\supset U_p$

(2) $(d\cdot\xi\zeta\eta)=(\zeta\eta,\xi)$ if ${}_R(\xi\zeta\eta)$ is a subduction $U_{p_1p_2}\supset U_{p_1}\times U_{p_2}$

(3) $(d\cdot\xi\zeta\eta)=(\zeta\eta\dagger,\xi)$ if ${}_R(\xi\zeta\eta)$ is a subduction $U_{q+\bar{q}}\supset U_q\times U_{\bar{q}}$

The complex conjugation and transposition symmetries relate pairs of triples. These will be denoted $(k\cdot\xi\zeta\eta)$ and $(\tau\cdot\xi\zeta\eta)$ respectively. For the unitary group ${}_R(k\cdot\xi\zeta\eta)$ $=(\xi^*\zeta^*\eta^*)={}_R(\bar{\xi}\bar{\zeta}\bar{\eta})$ where $\bar{\xi},\bar{\zeta},\bar{\eta}$ are the pure contravariant labels obtained from ξ,ζ,η. The special unitary group triple ${}_R(\xi'\zeta'\eta')$ has the complex conjugate triplet ${}_R(k\cdot\xi'\zeta'\eta')={}_R(\xi'^*\zeta'^*\eta'^*)$ which is labelled by pure covariant labels. The symmetric group has real irreps hence the symmetric group triple $(k\cdot\xi\zeta\eta)$ equals $(\xi\zeta\eta)$. The transposition symmetry relates any triple $(\xi\zeta\eta)$ with $(\tau\cdot\xi\zeta\eta)$ $=(\zeta\xi,\eta)$ or $(\xi,\eta\zeta)$ depending on the group-subgroup scheme considered. Using the above notation, we rewrite the phase freedom and symmetries of the duality factors (V.4.3, 4, 6, 9, 10) as

$$
\begin{aligned}
&\text{(i)} && \hat{D}_R(\xi\zeta\eta)=U[d\cdot\xi\zeta\eta]D_R(\xi\zeta\eta)U_R\{\xi\zeta\eta\}^\dagger\\
& && \hat{D}_R(\bar{\xi}\bar{\zeta}\bar{\eta})=U[d\cdot\xi\zeta\eta]D_R(\bar{\xi}\bar{\zeta}\bar{\eta})U_R\{\bar{\xi}\bar{\zeta}\bar{\eta}\}^\dagger\\
&\text{(ii)} && D_R(\bar{\xi}\bar{\zeta}\bar{\eta})=A[d\cdot\xi\zeta\eta]D_R(\xi\zeta\eta)^*A_R\{\xi\zeta\eta\}^{\dagger *}\\
& && D_R(\xi\zeta\eta)=A[d\cdot\xi\zeta\eta]D_R(\bar{\xi}\bar{\zeta}\bar{\eta})^*A_R\{\bar{\xi}\bar{\zeta}\bar{\eta}\}^{\dagger *}\\
&\text{(iii)} && D_R(\tau\cdot\xi\zeta\eta)=T[d\cdot\xi\zeta\eta]D_R(\xi\zeta\eta)T_R\{\xi\zeta\eta\}^\dagger\\
& && D_R(\tau\cdot\bar{\xi}\bar{\zeta}\bar{\eta})=T[d\cdot\xi\zeta\eta]D_R(\bar{\xi}\bar{\zeta}\bar{\eta})T_R\{\bar{\xi}\bar{\zeta}\bar{\eta}\}^\dagger
\end{aligned}
\tag{VI.3.3}
$$

In addition the phase freedom and symmetries of the isomorphism factors are rewritten as

$$
\begin{aligned}
&\text{(i)} && \hat{F}_R\{\xi\zeta\eta\} = U'_R\{\xi'\zeta'\eta'\}F_R\{\xi\zeta\eta\}U_R\{\xi\zeta\eta\}^\dagger\\
&\text{(ii)} && F_R\{k\cdot\xi\zeta\eta\} = A'_R\{\xi'\zeta'\eta'\}F_R\{\xi\zeta\eta\}^*A_R\{\xi\zeta\eta\}^{\dagger *}\\
&\text{(iii)} && F_R\{\tau\cdot\xi\zeta\eta\} = T'_R\{\xi'\zeta'\eta'\}F_R\{\xi\zeta\eta\}T_R\{\xi\zeta\eta\}^\dagger
\end{aligned}
\tag{VI.3.4}
$$

We have assumed in the coupling scheme that all the transformation factors of U_1 are unity and the U_1 phase freedom has been fixed.

We shall need the following terminology. Two triples $_R(\xi_1\zeta_1\eta_1)$ and $_R(\xi_2\zeta_2\eta_2)$ will be termed associated triples if each pair of partitions ξ_1 and ξ_2, ζ_1 and ζ_2, η_1 and η_2 are associated. Two partitions labelling irreps of U_p are associated if the one is obtained from the other by the addition of an integer, positive or negative, to each part of the p part partition. Notationally this is written $\{\mu\} = \{\lambda+\zeta^P\}$ where $\mu_i=\lambda_i+\zeta$ for $i=1,\dots p$. Associated triples are related in the same way with the requirement that the addition of the rectangular partitions must always be performed such that another triple is formed. The set of all associated triples can be formed by this "add" procedure.

If ξ,ζ,η are partitions of a,b,c respectively, we defined the lowest triple of a set of associated triples as the pure covariant triple for which a,b,c take the lowest values. This triple will be denoted $_R(\xi_0\zeta_0\eta_0)$. For fixed irreps ξ,ζ,η, there exist a "minimal group scheme" in which $_R(\xi\zeta\eta)$ exists, but for lower group schemes the triple does not exist. We denote this minimal scheme as R_0 and the corresponding triple by $_{R_0}(\xi\zeta\eta)$. Finally, if ξ',ζ',η' are partitions of a,b,c respectively, then, the special unitary group triple $_R(\xi'\zeta'\eta')$ is "stretched" if

(1) in the coupling scheme $SU_p\times SU_p\supset SU_p$, a+b=c

(2) in the subduction scheme $SU_{p_1p_2}\supset SU_{p_1}\times SU_{p_2}$, a=b=c

(3) in the subduction scheme $SU_{q+\bar{q}}\supset SU_q\times SU_{\bar{q}}$, a=b+c.

The triple is nonstretched if the condition does not hold.

We turn to the problem of the matrix choices for duality and isomorphism factors.

4. MATRIX CHOICES

In this section we discuss the possible matrix choices for the duality factors and isomorphism factors. We give two alternative procedures by which these factors can be chosen. The first involves choosing all isomorphism factors unity. This choice ensures the $U_p \simeq U_1 \times SU_p$ relationship is simple, and hence obtaining the algebraic formulae for U_p 6j symbols from the table of algebraic formulae of SU_p 6j symbols is easy (see Chapter IV). The second procedure is equivalent to previous authors' work and results in a unity choice for all duality factors. Both procedures, however, give rise to the same problem, that is an insufficient phase freedoms to choose either all the duality factors or all the isomorphism factors. As we will show this insufficiency implies that some duality/isomorphism factors cannot be determined by the relations presented in Section 3.

To begin we assume that complete phase freedom exists for the GH transformation theory of both the unitary group, special unitary group and symmetric group schemes. The first procedure requires a unity choice of all isomorphism factors. This is obtained by choosing in (3.4.i)

$$U_R\{\xi\zeta\eta\}=U'_R\{\xi'\zeta'\eta'\}F_R\{\xi\zeta\eta\} \qquad \text{(VI.4.1)}$$

for all pure covariant, pure contravarient and composite labelled triples. Hence, all isomorphism factors are unity

$$\hat{F}_R\{\xi\zeta\eta\} = I \qquad \text{(VI.4.2)}$$

with the phase freedom restriction

$$U_R\{\xi\zeta\eta\}=U_R'\{\xi'\zeta'\eta'\} \qquad \text{(VI.4.3)}$$

which implies that all unitary group phase freedoms are fixed with respect to special unitary group phase freedoms. In addition the unitary group complex conjugation and transposition factors are determined from the special unitary group factors.

$$A_R\{\xi\zeta\eta\}=A_R'\{\xi'\zeta'\eta'\} \qquad \text{(VI.4.4)}$$

$$T_R\{\xi\zeta\eta\}=T_R'\{\xi'\zeta'\eta'\} \qquad \text{(VI.4.5)}$$

Incorporating the restriction (4.3) into (3.3i), we have

$$\hat{D}_R(\xi\zeta\eta)=U[d\cdot\xi\zeta\eta]D_R(\xi\zeta\eta)U_R'\{\xi'\zeta'\eta'\}^\dagger \qquad \text{(VI.4.6)}$$

$$\hat{D}_R(\bar{\xi}\bar{\zeta}\bar{\eta})=U[d''\xi\zeta\eta]D_R(\bar{\xi}\bar{\zeta}\bar{\eta})U_R'\{\bar{\xi}'\bar{\zeta}'\bar{\eta}'\}^\dagger \qquad \text{(VI.4.7)}$$

We consider the pure covariant triples. Two cases arise:

(1) R is the minimal scheme for the triple $_R(\xi\zeta\eta)$. In obtaining the special unitary group triple $_R(\xi'\zeta'\eta')$, the triple $_R(\xi\zeta\eta)$ is always modified. We make the following choice of symmetric group phase freedom

$$U[d\cdot\xi\zeta\eta]=U_R'\{\xi'\zeta'\eta'\}D_R(\xi\zeta\eta)^\dagger \qquad \text{(VI.4.8)}$$

The corresponding duality factors are unity

$$\hat{D}_R(\xi\zeta\eta) = I \qquad \text{(VI.4.9)}$$

with the phase restriction

$$U[d\cdot\xi\zeta\eta]=U_R'\{\xi'\zeta'\eta'\} \qquad \text{(VI.4.10)}$$

(2) R is a non-minimal scheme, in which case the special unitary group triple is obtained from ${}_R(\xi\zeta\eta)$ unmodified, that is ${}_R(\xi'\zeta'\eta')={}_R(\xi\zeta\eta)$ for $R>R_0$. We may choose

$$U'_R\{\xi\zeta\eta\}=U[d\cdot\xi\zeta\eta]D_R(\xi\zeta\eta) \tag{VI.4.11}$$

with the result that

$$\hat{D}_R(\xi\zeta\eta) = I \tag{VI.4.12}$$

and the phase restriction

$$U'_R\{\xi\zeta\eta\}=U[d\cdot\xi\zeta\eta] \tag{VI.4.13}$$

The above gives the essential steps for a recursive procedure of choosing all pure covariant duality factors. This can be seen as follows. Using (1), we choose all duality factors $D_R(\xi\zeta\eta)$ corresponding to associate triples of the lowest triple ${}_R(\xi_0\zeta_0\eta_0)$. Note that ${}_R(\xi\zeta\eta)$ is a triple belonging to a minimal scheme but ${}_R(\xi_0\zeta_0\eta_0)$ may not necessarily be so. If the lowest triple $R(\xi_0\zeta_0\eta_0)$ belongs to a non-minimal scheme, the duality factor $D_R(\xi_0\zeta_0\eta_0)$ belongs to a set of duality factors labelled by the same partitions $\xi_0\zeta_0\eta_0$ but with varying R. With the exception of $D_{R_0}(\xi_0\zeta_0\eta_0)$ where R_0 is the minimal scheme of $(\xi_0\zeta_0\eta_0)$, the choices of this set of duality factors is given by (2). The procedure is repeated until the lowest triple belonging to the minimal scheme is reached. In this recursive procedure we have equated through (4.10) and (4.13) a certain infinite set of symmetric group and special unitary group phase freedoms to just one single special unitary group phase freedom factor $U'_{R_0}\{\xi'_0\zeta'_0\eta'_0\}$. To find this set it is a simple matter of taking the triple ${}_{R_0}(\xi_0\zeta_0\eta_0)$, forming all associate triples, and then increasing the unitary group ranks of the scheme R_0 and repeating the process to all triples formed in this manner.

The phase freedom factor $U'_{R_0}\{\xi'_0\zeta'_0\eta'_0\}$ is characterised by the fact that ${}_{R_0}(\xi'_0\zeta'_0\eta'_0)$ is a non-stretched triple. The set of all such factors is the only freedom left and it is insufficient to choose all pure contravariant duality factors. The phase freedom given by (3.3.i) is restricted to

$$\begin{aligned}D_R(\bar{\xi}\bar{\zeta}\bar{\eta})&=U[d\cdot\xi\zeta\eta]D_R(\bar{\xi}\bar{\zeta}\bar{\eta})U_R\{\bar{\xi}\bar{\zeta}\bar{\eta}\}^\dagger\\&=U'_{R_0}\{\xi'_0\zeta'_0\eta'_0\}D_R(\bar{\xi}\bar{\zeta}\bar{\eta})U'_R\{\bar{\xi}'\bar{\zeta}'\bar{\eta}'\}^\dagger\end{aligned}\tag{VI.4.14}$$

where ${}_R(\bar{\xi}'\bar{\zeta}'\bar{\eta}')$ is the special unitary group triple obtained from ${}_R(\bar{\xi}\bar{\zeta}\bar{\eta})$ by association and ${}_{R_0}(\xi'_0\zeta'_0\eta'_0)$ is the non-stretched triple obtained from ${}_R(\xi\zeta\eta)$ by repeated application of (4.10) and (4.13). By association there is an infinite number of duality factors $D_R(\bar{\xi}\bar{\zeta}\bar{\eta})$ which give rise to the same phase freedom factors $U'_R\{\bar{\xi}'\bar{\zeta}'\bar{\eta}'\}$ and $U'_{R_0}\{\xi'_0\zeta'_0\eta'_0\}$. Thus not all pure contravariant duality factors can be chosen.

The above unit choice of the pure covariant duality factors simplifies (3.3) as follows

$$A_R\{\xi\zeta\eta\}=A[d\cdot\xi\zeta\eta]\tag{VI.4.15}$$

and

$$T_R\{\xi\zeta\eta\}=T[d\cdot\xi\zeta\eta]\quad\text{for all } R\tag{VI.4.16}$$

Together with (4.4-5), these relations equate all complex conjugation (and transposition) factors of the unitary group, special unitary group and symmetric group to a special unitary group complex conjugation (respectively transposition) factor labelled by a non-stretched triple. It is these factors which must be determined subject to all the above phase freedom restrictions (4.3, 4.10, 4.13). Different choices from those given in Chapter III will occur.

In the second procedure we first choose both pure covariant and pure contravariant duality factors unity. This can be done by choosing

$$U_R\{\xi\zeta\eta\}=U[d\cdot\xi\zeta\eta]D_R(\xi\zeta\eta) \tag{VI.4.17}$$

for all pure covariant triples $_R(\xi\zeta\eta)$ and

$$U_R\{\bar{\xi}\bar{\zeta}\bar{\eta}\}=U[d\cdot\xi\zeta\eta]D_R(\bar{\xi}\bar{\zeta}\bar{\eta}) \tag{VI.4.18}$$

for all pure contravariant triples $_R(\xi\zeta\eta)$. Hence

$$\hat{D}_R(\xi\zeta\eta) = I \tag{VI.4.19}$$

and

$$\hat{D}_R(\bar{\xi}\bar{\zeta}\bar{\eta}) = I \tag{VI.4.20}$$

with the phase freedom restriction

$$U_R\{\xi\zeta\eta\}=U\{\bar{\xi}\bar{\zeta}\bar{\eta}\}=U[d\cdot\xi\zeta\eta]. \tag{VI.4.21}$$

Such choices relate quite simply all unitary group transformation factors to symmetric group factors. For example, (3.3.ii - iii) give

$$A_R\{\xi\zeta\eta\}=A_R\{\bar{\xi}\bar{\zeta}\bar{\eta}\}=A[d\cdot\xi\zeta\eta] \tag{VI.4.22}$$

and

$$T_R\{\xi\zeta\eta\}=T_R\{\bar{\xi}\bar{\zeta}\bar{\eta}\}=T[d\cdot\xi\zeta\eta] \tag{VI.4.23}$$

The phase freedom restriction (4.21) impose a constraint on the isomorphism factors

(i) for pure covariant triples

$$\hat{F}_R\{\xi\zeta\eta\}=U_R'\{\xi'\zeta'\eta'\}F_R\{\xi\zeta\eta\}U[d\cdot\xi\zeta\eta]^{\dagger} \tag{VI.4.24}$$

(ii) for pure contravariant triples

$$\hat{F}_R\{\bar{\xi}\bar{\zeta}\bar{\eta}\}=U'_R\{\bar{\xi}'\bar{\zeta}'\bar{\eta}'\}F_R\{\bar{\xi}\bar{\zeta}\bar{\eta}\}U[d\cdot\xi\zeta\eta]^{\dagger} \qquad \text{(VI.4.25)}$$

The phase freedom of the isomorphism factors is identical to that of the duality factors in (4.6-7). The same arguments for the duality factors apply in establishing matrix choices for the isomorphism factors. We have for pure covariant triples

(1) if R is a minimal scheme for the triple ${}_R(\xi\zeta\eta)$, then let

$$U[d\cdot\xi\zeta\eta]=U'_R\{\xi'\zeta'\eta'\}F_R\{\xi\zeta\eta\} \ . \qquad \text{(VI.4.26)}$$

Hence $\hat{F}_R\{\xi\zeta\eta\} = I$ (VI.4.27)

and $U[d\cdot\xi\zeta\eta]=U'_R\{\xi'\zeta'\eta'\}$ (VI.4.28)

(2) if R is a non-minimal scheme, ${}_R(\xi'\zeta'\eta')={}_R(\xi\zeta\eta)$ and we choose

$$U'_R\{\xi\zeta\eta\}=U[d\cdot\xi\zeta\eta]F_R\{\xi\zeta\eta\}^{\dagger} \qquad \text{(VI.4.29)}$$

which gives $\hat{F}_R\{\xi\zeta\eta\} = I$ (VI.4.30)

and

$$U'_R\{\xi\zeta\eta\}=U[d\cdot\xi\zeta\eta] \ . \qquad \text{(VI.4.31)}$$

The phase freedom for pure contravariant isomorphism factors is restricted by (4.28) and (4.31)

$$\hat{F}_R\{\xi\zeta\eta\}=U'_R\{\bar{\xi}'\bar{\zeta}'\bar{\eta}'\}F_R\{\bar{\xi}\bar{\zeta}\bar{\eta}\}U'_{R_o}\{\xi'_o\zeta'_o\eta'_o\}^{\dagger} \ . \qquad \text{(VI.4.31)}$$

For all associated pure contravariant triples of ${}_R(\bar{\xi}\bar{\zeta}\bar{\eta})$, the corresponding isomorphism factor gives rise to the same special unitary group phase freedom factors $U'_R\{\bar{\xi}'\bar{\zeta}'\bar{\eta}'\}$ and $U'_{R_0}\{\bar{\xi}'\bar{\zeta}'\bar{\eta}'\}$. Hence there is insufficient phase freedom to choose all these associated pure contravariant isomorphism factors. We note that the composite labelled unitary group phase freedom is not contrained by duality. We can therefore always make a unity choice for the composite labelled isomorphism factors.

CHAPTER VII

CONCLUDING REMARKS

In this thesis we have studied three strongly interrelated topics with the main aim of evaluating 6j and 3jm symbols of the unitary group. We have developed the general group-subgroup transformation theory giving a new perspective to group-subgroup reduction theory particularly the Racah-Wigner coupling algebra. We have defined new transformation factors arising from coupled, subduced and induced representation spaces. Various types of transformation factors, such as the coupling, subduction and induction factors and the recoupling, resubduction and reinduction factors, are found to display similar symmetry properties. These similarities have been used in proposing a method of calculation, similar to that for 6j and 3jm symbols, for the different types of transformation factors. Such a calculation has not been performed but from the success of the Butler method for 6j and 3jm symbols, there would appear to be no real difficulties. The exception would be the orientation phase problem which appears to be related to the way subgroups are imbedded in a group. Why this problem appears in some embeddings and not in others is largely unresolved.

The Butler method was employed to obtain algebraic formulae for 6j symbols of the special unitary group up to and including power 3 primitives. These formulae agreed with our earlier numerical tables of SU_3 and SU_6 up to phase freedom choices. We have used the composite labelling, a non-standard labelling, in giving the table of SU_n 6j symbols. This labelling has some advantages in that the 6j symbol immediately displays a new symmetry - the transpose conjugate symmetry which originates from the one-dimensional alternating irrep of the symmetric group. Composite label modification rules in U_n also give partial answers to why certain factors appear in the 6j formulae. This line of work gives credence to the idea that an algebraic formulae may be obtained for a 6j symbol from its irrep content and combinatoric properties.

Many lines of research are suggested, in particular using the Butler method to calculate algebraic formulae for 3jm symbols of the special unitary group schemes $SU_{mn} \supset SU_m \times SU_n$ and $SU_{m+n} \supset SU_m \times SU_n$ and the 6j and 3jm of the other Lie groups O_n, Sp_n and SO_n. All these groups are of current interest in physical calculations. Modification rules for the labels of irreps for other Lie groups may prove to be useful in determining the form of the algebraic formulae of the respective 6j and 3jm symbols.

In the final chapters, we have used the GH transformation theory to describe the Schur-Weyl duality relation between the symmetric group and the unitary group. We have developed five relations between symmetric group transformation factors and unitary group transformation factors. Each involves the appearance of duality factors. We have shown in connection with the $U_n \simeq U_1 \times SU_n$ isomorphism symmetry, that many of the duality and isomorphism factors can be chosen the unit matrix, however there is insufficient freedom to choose all of them. Whether these factors can also be determined the unit matrix may be answered by an extension of the Schur-Weyl duality. This extension takes the form of relating the symmetric group with an arbitrary group via the resolution of the f-Kronecker product of a representation of a group into its symmetrised parts. Similar relations to those of the Schur-Weyl duality can be obtained with duality factors suitably defined. Taking the special unitary group as our arbitrary group we have a more direct way of relating the symmetric group with the special unitary group. The S_f-SU_p duality factors can then be used to determine the remaining S_f-U_p duality factors using the $U_p \simeq U_1 \times SU_p$ isomorphism symmetry.

APPENDIX

3 jm and 6j Tables for some bases of SU_6 and SU_3

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$3jm$ and $6j$ tables for some bases of SU_6 and SU_3

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Abstract. Tables of $6j$ symbols for SU_6 and SU_3 and tables of $3jm$ factors for $SU_6 \supset SU_2 \times SU_3$, $SU_3 \supset U_1 \times SU_2$ and $SU_3 \supset SO_3$ are presented. The tables are computer produced, using a program that implements the building up principle in a general form. Our tables are useful for calculations in high energy, nuclear and solid state physics. Some other tabulations contain errors, and none uses all the symmetries available. The n independence of our SU_n results is discussed by using the various symmetric group–unitary group duality relations.

1. Introduction

It was recognised long ago (Wigner 1931) that the quantum theory of angular momentum is intimately related to the three-dimensional rotation group. Many aspects of this theory may be generalised to other groups. Griffith (1961) and Koster *et al* (1963) calculated coupling coefficients for point groups, which have many applications in solid state and molecular physics calculations. Racah's (1949) work on fractional parentage coefficients (CFP) showed that groups larger than SO_3 are useful for classifying and constructing atomic states, and much use has been made of these methods in nuclear theory. More recently SU_3 and other unitary groups have been applied to hadron model calculations.

The highly symmetric $3jm$ and $6j$ symbols introduced by Wigner (1940) have many advantages over the less symmetric coupling and recoupling coefficients calculated by Griffith (1961) and Koster *et al* (1963) for point groups, de Swart (1963) for SU_3 and Cook and Murtaza (1965), Schülke (1965) and Carter *et al* (1965) for SU_6. The calculation of the coefficients is greatly simplified, the tabulation is much more compact and the various applications involve the use of more symmetrical equations. $3jm$'s and $6j$'s can be defined for any group (Derome and Sharp 1965, Butler 1975) but many authors still calculate unsymmetrised coefficients, for example Akiyama and Draayer (1973) and Draayer and Akiyama (1973) for SU_3, and Machacek and Tomozawa (1976) and So and Strottman (1979) for SU_6.

Several methods have been used to calculate $3jm$'s and $6j$'s (or coupling and recoupling coefficients). These include projection-operator construction of states, which often involves a transformation from a known basis (such as Akiyama and Draayer's construction of the $SU_3 \supset SO_3$ basis from the $SU_3 \supset U_1 \times SU_2$ basis), ladder operator techniques, which make use of matrix elements and generators (the usual angular momentum approach—see Louck and Biedenharn (1973) for U_n and Alisauskas and Norvaisas (1980) for $SU_n \supset SO_n$) and the building-up method we use here. The

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building-up process was shown by Butler and Wybourne (1976) to have the advantage of requiring only a knowledge of character theory, chiefly product and branching rules. This is particularly useful for groups with irreps of large dimension such as E_7 (Butler *et al* 1978, 1979), where the other methods are impractical because they involve the construction of very large matrices.

The unitary groups, especially SU_3 and SU_6, have been used extensively since Jahn's (1950) extension of Racah's (1949) CFP work. Harmonic oscillator calculations involve SU_3 (Wybourne 1974, ch 20), and SU_3 and SU_6 occur in particle physics, particularly in the application of the Wigner–Eckart theorem to the colour hyperfine interaction. Some of the tables presented here have been used by Bickerstaff and Wybourne (1980, 1981) and Black and Wybourne (1981) for multiquark hadron calculations. Likewise, crystal field calculations in the 'strong-field' parentage scheme (Tanabe and Sugano 1954a, b, 1956, Kustov 1977) use the $SU_6 \supset SU_2 \times SU_3$ and $SU_3 \supset SO_3$ bases.

Some previous tabulations contain errors and none displays all known symmetries. Further, there is the vexing problem of consistency between various tables (for some relevant comments see Bickerstaff (1982)). We present tabulations of $6j$ symbols for SU_3 and SU_6 and $3jm$ factors for $SU_6 \supset SU_2 \times SU_3$, $SU_3 \supset U_1 \times SU_2$ and $SU_3 \supset SO_3$ which are consistent with our $6j$ tables and the standard SU_2 tables of Rotenberg *et al* (1959). So and Strottman's (1979) tables for U_6 are not applicable to SU_6 because their phase prescription results in different phases for coefficients which are equivalent under SU_6, while Hogaasen and Sorba (1978) contains normalisation errors. Kustov's (1977) SU_6 and SU_3 tables contain errors because he has overlooked the question of product multiplicity in SU_3. Our values are given as exact complex numbers rather than the floating-point values used in Akiyama and Draayer's (1973) calculations for SU_3.

2. The building-up method of calculation

The theory of the generalised Racah–Wigner algebra has been given by Derome and Sharp (1965) and Butler (1975). Butler and Wybourne (1976) recognised that sufficient recursion relations exist in the algebra to allow the calculation of $6j$ symbols and then $3jm$ factors, up to the phase and multiplicity freedoms allowed by Schur's lemmas. This building-up process was used by Butler (1981) to generate tables for all point groups.

The continuous groups we consider here have an infinite number of irreps. In the absence of closed analytic formulae (such as Racah's (1942) formulae for SO_3 and $SO_3 \supset SO_2$) we are restricted to calculating a finite number of $6j$ symbols and $3jm$ factors. In deciding which symbols to include, the concept of power is used. We choose a faithful irrep, ε (the irrep $\{1\}$ for SU_3 and SU_6), and call it the primitive. The power of λ is defined to be the smallest p for which $(\varepsilon + \varepsilon^*)^p \supset \lambda$. The building-up method may be used to calculate all $6j$ symbols and $3jm$ factors up to a certain power: our tables include all irreps up to power 3 in SU_3 and SU_6.

All the groups we consider are quasi-ambivalent and no non-simple phase irreps occur in the $6j$'s and $3jm$'s we have calculated (see Butler (1975) for a definition of these terms). SU_3 has no non-simple phase irreps (Derome 1967) and the first known non-simple phase irrep of SU_6 is $\{432^21\}$ which is power 6 (Butler and King 1974). Because of these simplifications the equations and the symmetries of Butler (1975) may be simplified to those of the point groups (see Butler (1981) and § 3). We stress that tables which fail, via their phase choices, to employ all possible symmetries require more complicated equations. Indeed, compare the expression for a $9j$ symbol as a sum of $6j$ symbols (Butler 1981, equation (3.3.37)) with Millener's (1978) calculations of unsymmetrised coefficients of SU_3. The choice that Derome and Sharp's (1965) A matrix is unity, forces some $6j$ and $3jm$ values to be imaginary. It is known that even without this choice some $3jm$ factors for some imbeddings are always complex. Butler (1980, 1981) gives a simple example for $T \supset D_2$.

Butler (1981) and Bickerstaff and Wybourne (1981) have given a detailed account of the phase and multiplicity freedoms in the $6j$ symbols and $3jm$ factors. Phase choices analogous to Reid and Butler's (1980) 'orientation' choices occur in each of the $3jm$ factor tables presented here. The distinguishing feature of an orientation choice is that the transformation between two sets of $3jm$ factors with different orientation choices requires primitive transformation factors which are not unity. For point groups this restriction is equivalent to a rotation of the axes.

Some choices of multiplicity separation lead to large prime numbers appearing in the tables. Our choices are based on the *ad hoc* requirement that numbers should be as simple as possible. Butler and Ford (1979) used a special symmetry of the octahedral group to simplify the numerical values of the $6j$ symbols for that group, but no such symmetries exist for SU_n.

3. Guide to the tables

The $6j$ and $3jm$ tables were calculated by the computer program which generated the point group tables of Butler (1981). The tables are computer typeset to preserve accuracy.

Schur-function notation (namely partitions) is used to label irreps (Wybourne 1970, 1974). For U_1 both positive and negative integers occur. The spin covering group of SO_3 is SU_2 and so one has a choice of labelling: either integers and half-integers, j, for SO_3, or integers k for SU_2, where $k = 2j$. The $6j$ symbols of SO_3 (and therefore for SU_2) are tabulated by Rotenberg *et al* (1959). They include the $3jm$ factors for $SO_3 \supset SO_2$ ($SU_2 \supset U_1$) under the name '$3j$ symbols'. For our table $SU_3 \supset SO_3$ we use the SO_3 labels, while for $SU_3 \supset U_1 \times SU_2$ and $SU_6 \supset SU_2 \times SU_3$ we use the SU_2 labels.

Table 1 lists the irreps of SU_3, up to power 3, giving complex conjugation properties and dimensions. Table 2 contains similar information for SU_6.

The Kronecker product rules are specified by means of triads. If $\lambda_1 \times \lambda_2 \supset \lambda_3^*$ then $\lambda_1 \times \lambda_2 \times \lambda_3 \supset 0$, where 0 is the identity irrep and * denotes complex conjugation. A triad is the set of three irrep labels, λ_1, λ_2 and λ_3, together with a multiplicity index r. The r labels any multiple occurrence of the identity irrep. The triad structure of SO_3 is well known, but the SO_2 structure of the '$3j$ symbols' is often ignored: if three irreps $m_1\ m_2\ m_3$ of U_1 (or SO_2) are to form a triad (multiplicity zero only) then $m_1 + m_2 + m_3 = 0$. All $6j$ symbols of U_1 (and SO_2) are chosen to be $+1$. Each triad has an associated phase, the $3j$ phase, which gives the symmetry on reordering coupled products. All

Table 1. Irreps of SU_3.

Irrep	0	1	1^2	2	2^2	21	3	3^2	31	32
Complex conjugate	0	1^2	1	2^2	2	21	3^2	3	32	31
Dimension	1	3	3	6	6	8	10	10	15	15

Table 2. Irreps of SU_6.

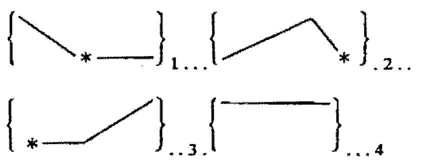
Irrep	0	1	1^5	1^2	1^4	2	2^5	21^4	1^3
Complex conjugate	0	1^5	1	1^4	1^2	2^5	2	21^4	1^3
Dimension	1	6	6	15	15	21	21	35	20
Irrep	21	2^41	21^3	2^21^3	3	3^5	31^4	32^4	
Complex conjugate	2^41	21	2^21^3	21^3	3^5	3	32^4	31^4	
Dimension	70	70	84	84	56	56	120	120	

relevant $3j$'s $\{\lambda_1\, \lambda_2\, \lambda_3\, r\}$ of SU_3 and SU_6 are tabulated in tables 3 and 5 respectively. For SO_3 one has $\{j_1\, j_2\, j_3\, 0\} = (-1)^{j_1+j_2+j_3}$.

The $6j$ symbol is related to recouplings between a set of six irreps, by means of four couplings. The triads occur in the $6j$

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1 r_2 r_3 r_4}$$

in the order



that is, $\{\lambda_1 \mu_2^* \mu_3 r_1\}$, $\{\mu_1 \lambda_2 \mu_3^* r_2\}$, $\{\mu_1^* \mu_2 \lambda_3 r_3\}$, $\{\lambda_1 \lambda_2 \lambda_3 r_4\}$.

Symmetries are used to reduce the size of the tables. The full symmetries are given by Butler (1981) and Butler and Wybourne (1976), but to find a $6j$ in the tables one only needs the following. The $6j$ symbols are invariant under even permutations of the columns; there is the complex conjugation symmetry

$$\begin{Bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \end{Bmatrix}_{r_1 r_2 r_3 r_4} = \begin{Bmatrix} \lambda_1^* & \lambda_2^* & \lambda_3^* \\ \mu_1^* & \mu_2^* & \mu_3^* \end{Bmatrix}^*_{r_1 r_2 r_3 r_4}, \tag{3.1}$$

the row flip symmetries, the (23) flip being

$$= \begin{Bmatrix} \lambda_1^* & \mu_2 & \mu_3^* \\ \mu_1^* & \lambda_2 & \lambda_3^* \end{Bmatrix}_{r_4 r_3 r_2 r_1}, \tag{3.2}$$

and the column interchange symmetries, the (12) operation being

$$= \pm \begin{Bmatrix} \lambda_2 & \lambda_1 & \lambda_3 \\ \mu_2^* & \mu_1^* & \mu_3^* \end{Bmatrix}_{r_2 r_1 r_3 r_4}. \tag{3.3}$$

The sign $\pm$ is the same for all interchanges and is given in the tables immediately after the multiplicity indices.

The $6j$ symbols of SU_3 and SU_6 are tabulated in tables 4 and 6. The bold typeface headings denote the top line of the $6j$ symbol and each subsequent entry denotes a lower line (three irreps, four multiplicity labels), the interchange sign, and the value.

The branching rules are given in tables 7, 9 and 11. No branching multiplicity occurs for the cases considered. The tables include the sign of the corresponding $2jm$. This is always $+1$ for $SU_3 \supset SO_3$ and for $SU_6 \supset SU_2 \times SU_3$, while for $SU_3 \supset U_1 \times SU_2$ the

Table 3. $3j$ symbols of SU_3.

0 0 0 0+	21 2^2 2 0+	3^2 3 21 0+	31 31 1 0−
1 1 1 0−	21 21 0 0+	31 2^2 1 0+	31 31 2^2 0+
1^2 1 0 0+	21 21 21 0+	31 2^2 2^2 0−	31 31 31 0−
2 1^2 1^2 0+	21 21 21 1−	31 21 1^2 0+	31 31 31 1+
2 2 2 0+	3 2^2 1^2 0+	31 21 2 0−	32 31 0 0+
2^2 2 0 0+	3 21 21 0−	31 3 2 0+	32 31 21 0+
21 1^2 1 0+	3 3 3 0−	31 3^2 1^2 0−	32 31 21 1−
21 2 1 0−	3^2 3 0 0+	31 3^2 2 0+	32 31 3 0−

Table 4. $6j$ symbols of SU_3.

0	**0**	**0**		
0	0	0	0000+	$+1$
1	**1**	**1**		
1^2	1	0	0000+	$-1/3$
1^2	**1**	**0**		
0	0	1	0000+	$+1/\sqrt{3}$
1	1	1^2	0000+	$-1/3$
1^2	1^2	0	0000+	$+1/3$
2	**1^2**	**1^2**		
0	1	1^2	0000+	$+1/3$
2	**2**	**2**		
2^2	2	0	0000+	$+1/2.3$
2^2	**2**	**0**		
0	0	2	0000+	$+1/\sqrt{2.3}$
1^2	1^2	1	0000+	$+1/3\sqrt{2}$
2	2	2^2	0000+	$+1/2.3$
2^2	2^2	0	0000+	$+1/2.3$
21	**1^2**	**1**		
0	1^2	1^2	0000+	$+1/3$
1^2	1	1	0000+	$+1/2.3$
2	1	1	0000+	$+1/4.3$
21	1^2	1^2	0000+	$-1/8.3$
21	**2**	**1**		
0	1^2	2	0000+	$-1/3\sqrt{2}$
1^2	1	1	0000+	$+1/2\sqrt{2.3}$
1^2	2^2	1	0000+	$-1/4.3$
2	1	2^2	0000+	$+1/4\sqrt{3}$
21	1^2	1^2	0000+	$-1/8$
21	1^2	2	0000+	$+\sqrt{5}/8.3$
21	2	1^2	0000+	$+1/8$
21	**2^2**	**2**		
0	2^2	2^2	0000+	$+1/2.3$
1	1^2	1^2	0000+	$+\sqrt{5}/4.3$
2^2	2	2	0000+	$-1/4.3$
21	1	1	0000+	$+\sqrt{5}/8.3$
21	1	2^2	0000+	$-1/8$
21	2^2	1	0000+	$-1/8$
21	2^2	2^2	0000+	$+11/8.3.5$
21	**21**	**0**		
0	0	21	0000+	$+1/2\sqrt{2}$
21	**21**	**0**		
1	1	1	0000+	$+1/2\sqrt{2.3}$
1	1	2^2	0000+	$-1/2\sqrt{2.3}$
1^2	1^2	1^2	0000+	$+1/2\sqrt{2.3}$
1^2	1^2	2	0000+	$-1/2\sqrt{2.3}$
2	2	1^2	0000+	$-1/4\sqrt{3}$
2	2	2	0000+	$+1/4\sqrt{3}$
2^2	2^2	1	0000+	$-1/4\sqrt{3}$
2^2	2^2	2^2	0000+	$+1/4\sqrt{3}$
21	21	0	0000+	$+1/8$
21	**21**	**21**		
1	1	1	0000+	$+\sqrt{5}/8\sqrt{2.3}$
1	1	1	0001−	$+i\sqrt{3}/8\sqrt{2}$
1^2	1^2	1^2	0000+	$+\sqrt{5}/8\sqrt{2.3}$
1^2	1^2	1^2	0001−	$-i\sqrt{3}/8\sqrt{2}$
2	1^2	1^2	0000+	$+\sqrt{5}/8\sqrt{2.3}$
2	1^2	1^2	0001−	$+i/8\sqrt{2.3}$
2	2	1^2	0000+	$-1/8\sqrt{2.3}$
2	2	1^2	0001−	$-i\sqrt{5}/8\sqrt{2.3}$
2	2	2	0000+	$+7/8.5\sqrt{2.3}$
2	2	2	0001−	$+i\sqrt{3}/8\sqrt{2.5}$
2^2	1	1	0000+	$+\sqrt{5}/8\sqrt{2.3}$
2^2	1	1	0001−	$-i/8\sqrt{2.3}$
2^2	2^2	1	0000+	$-1/8\sqrt{2.3}$
2^2	2^2	1	0001−	$+i\sqrt{5}/8\sqrt{2.3}$
2^2	2^2	2^2	0000+	$+7/8.5\sqrt{2.3}$
2^2	2^2	2^2	0001−	$-i\sqrt{3}/8\sqrt{2.5}$
21	21	0	0000+	$+1/8$
21	21	0	0001−	0
21	21	0	0010−	0
21	21	0	0011+	$-1/8$
21	21	21	0000+	$-3/8.2.5$
21	21	21	0001−	0
21	21	21	1000−	0
21	21	21	1001+	$-1/8.2$
21	21	21	1100+	$-1/8.2$
21	21	21	1101−	0
21	21	21	1110−	0
21	21	21	1111+	$+1/8.2$
3	**2^2**	**1^2**		
0	1	2^2	0000+	$+1/3\sqrt{2}$
1	2	1^2	0000+	$+1/2.3$
21	1	2^2	0000+	$+1/2.3\sqrt{5}$
3	**21**	**21**		
0	21	21	0000+	$-1/8$
1^2	2	1^2	0000+	$+1/4\sqrt{3}$
2	2	1^2	0000+	$+1/4\sqrt{3.5}$
21	21	21	0000+	$+1/4.5$
21	21	21	0100−	$-i/8\sqrt{5}$
21	21	21	0110+	0
3	21	21	0000+	$+1/8.5$
3^2	21	21	0000+	$+1/8.5$
3	**3**	**3**		
21	21	21	0000+	$-1/2.5\sqrt{2}$
3^2	3	0	0000+	$-1/2.5$
3^2	3	21	0000+	$+1/4.5$
3^2	**3**	**0**		
0	0	3	0000+	$+1/\sqrt{2.5}$
1^2	1^2	2	0000+	$+1/\sqrt{2.3.5}$
2^2	2^2	1	0000+	$+1/2\sqrt{3.5}$
21	21	21	0000+	$-1/4\sqrt{5}$
21	21	3	0000+	$+1/4\sqrt{5}$
3	3	3^2	0000+	$-1/4\sqrt{5}$
3^2	3^2	0	0000+	$+1/2.5$
3^2	**3**	**21**		
1^2	1^2	2	0000+	$+1/2\sqrt{3.5}$
2^2	2^2	1	0000+	$+1/5\sqrt{3}$
21	0	3	0000+	$+1/4\sqrt{5}$
21	21	21	0000+	$-1/4.5\sqrt{2}$
21	21	21	0010−	$+i/4\sqrt{2.5}$
21	21	3	0000+	$+3/8.5\sqrt{2}$
21	21	3	0010−	$-i/8\sqrt{2.5}$
3	3	3^2	0000+	$+1/4.5$
3^2	21	21	0000+	$-1/4.5$
3^2	3^2	0	0000+	$+1/2.5$
3^2	3^2	21	0000+	$+3/8.5$
31	**2^2**	**1**		
0	1^2	2^2	0000+	$+1/3\sqrt{2}$
21	1^2	2^2	0000+	$-1/4.3\sqrt{5}$

Table 4—*continued*

31	**2^2**	**1**		
21	2	1	0000+	$+1/4.3$
21	2	2^2	0000+	$-1/2\sqrt{2.3.5}$
3	2	1	0000+	$+1/2.3.5$
31	**2^2**	**2^2**		
0	2	2^2	0000+	$-1/2.3$
21	2	1	0000+	$-1/2\sqrt{2.3.5}$
21	2	2^2	0000+	$+1/2.3.5$
31	**21**	**1^2**		
0	1	21	0000+	$+1/2\sqrt{2.3}$
1	1^2	2^2	0000+	$+1/2.3\sqrt{2}$
1	2	1	0000+	$-1/4\sqrt{3}$
1	2	2^2	0000+	$+1/2.3\sqrt{2}$
1^2	21	1^2	0000+	$+1/8$
1^2	21	2	0000+	$+1/8$
2	21	1^2	0000+	$+1/8.3$
2	21	2	0000+	$+1/8\sqrt{5}$
2^2	1^2	2^2	0000+	$-1/4\sqrt{3.5}$
21	1	21	0000+	$+1/8\sqrt{2.3.5}$
21	1	21	0100−	$+i/8\sqrt{2.3}$
21	2^2	21	0000+	$+\sqrt{3}/8\sqrt{2.5}$
21	2^2	21	0100−	$-i/8\sqrt{2.3}$
3^2	2^2	21	0000+	$-1/4.5\sqrt{3}$
31	2	1	0000+	$-1/4.3.5$
31	2	2^2	0000+	$+1/2.5\sqrt{2.3}$
32	21	1^2	0000+	$+1/8.5$
31	**21**	**2**		
0	2^2	21	0000+	$-1/4\sqrt{3}$
1	1^2	2^2	0000+	$-1/4\sqrt{3}$
1^2	21	1^2	0000+	$+1/8$
1^2	21	2	0000+	$-1/8.3$
2	21	1^2	0000+	$+1/8\sqrt{5}$
2	21	2	0000+	$-1/8.5$
2^2	2	1	0000+	$-1/4.3$
2^2	2	2^2	0000+	$+1/2\sqrt{2.3.5}$
21	1	21	0000+	$+\sqrt{3}/8\sqrt{2.5}$
21	1	21	0100−	$-i/8\sqrt{2.3}$
21	2^2	21	0000+	$+7/8.5\sqrt{2.3}$
21	2^2	21	0100−	$-i/8\sqrt{2.3.5}$
3	1	21	0000+	$-1/4.5\sqrt{3}$
31	1^2	2^2	0000+	$+1/4.3.5$
31	2	1	0000+	$+1/2.5\sqrt{2.3}$
31	2	2^2	0000+	$+1/2.3.5$
32	21	1^2	0000+	$+1/8.3.5$
32	21	2	0000+	$+7/8.3.5$
31	**3**	**2**		
0	2^2	3	0000+	$+1/2\sqrt{3.5}$
1^2	21	2	0000+	$+1/3\sqrt{2.5}$
1^2	3^2	2	0000+	$+1/2.3.5$
2^2	2	1	0000+	$+1/3\sqrt{2.5}$
21	1	21	0000+	$+1/2\sqrt{2.3.5}$
21	2^2	21	0000+	$-1/2.5\sqrt{2.3}$
21	2^2	3	0000+	$-1/2.5\sqrt{3}$
3	1	3^2	0000+	$+1/5\sqrt{2.3}$
31	1^2	2^2	0000+	$+1/3.5$
31	2	2^2	0000+	$-1/3.5$

31	**3**	**2**		
32	21	1^2	0000+	$-1/3.5$
32	21	2	0000+	$-1/3.5\sqrt{2}$
32	3	1^2	0000+	$-1/3.5$
32	3	2	0000+	$+1/3.5$
32	3^2	2	0000+	$-1/2.3.5$
31	**3^2**	**1^2**		
0	1	3^2	0000+	$-1/\sqrt{2.3.5}$
1	1^2	2^2	0000+	$+1/3\sqrt{5}$
1	2	2^2	0000+	$-1/2.3\sqrt{5}$
2	21	1^2	0000+	$+1/3\sqrt{2.5}$
2	3	1^2	0000+	$-1/2.3.5$
21	1	21	0000+	$-1/2\sqrt{2.3.5}$
21	1	3^2	0000+	$+1/4\sqrt{3.5}$
21	2^2	21	0000+	$-1/2\sqrt{2.3.5}$
21	2^2	3^2	0000+	$-1/2\sqrt{2.3.5}$
3^2	2^2	3	0000+	$+1/5\sqrt{2.3}$
31	2	1	0000+	$-1/3.5$
31	2	2^2	0000+	$+1/2.5\sqrt{3}$
32	21	2	0000+	$-1/3.5$
31	**3^2**	**2**		
0	2^2	3^2	0000+	$+1/2\sqrt{3.5}$
1	1^2	2^2	0000+	$+1/2\sqrt{3.5}$
2	21	1^2	0000+	$-1/2.5$
21	1	21	0000+	$+1/4\sqrt{3.5}$
21	1	3^2	0000+	$-1/2\sqrt{2.3.5}$
21	2^2	21	0000+	$+\sqrt{3}/4.5$
21	2^2	3^2	0000+	$+1/2.5\sqrt{3}$
3	1	21	0000+	$-1/2.5\sqrt{3}$
31	1^2	2^2	0000+	$-1/2.3.5$
31	2	1	0000+	$+1/2.5\sqrt{3}$
31	2	2^2	0000+	$-1/3.5$
32	21	2	0000+	$-1/3.5\sqrt{2}$
32	3	2	0000+	$-1/2.3.5$
32	3^2	2	0000+	$+1/3.5$
31	**31**	**1**		
1	0	32	0000+	$-1/3\sqrt{5}$
1^2	1	21	0000+	$+1/2\sqrt{3.5}$
2	1	21	0000+	$+1/2.3\sqrt{2.5}$
2	1	3^2	0000+	$+1/3\sqrt{2.5}$
21	1^2	2^2	0000+	$-1/2\sqrt{2.3.5}$
21	1^2	31	0000+	$+1/2.3\sqrt{5}$
21	1^2	31	0100−	0
21	2	1	0000+	$+1/2\sqrt{2.3.5}$
21	2	2^2	0000+	$+1/2.3\sqrt{5}$
21	2	31	0000+	$-1/2.3\sqrt{2.5}$
21	2	31	0100−	$+i/3\sqrt{2.3.5}$
3	2	2^2	0000+	$+1/3.5\sqrt{2}$
3	2	31	0000+	$+1/3.5\sqrt{3}$
31	21	1^2	0000+	$-1/2.3.5$
31	21	2	0000+	$-1/2.3.5\sqrt{2}$
31	21	32	0000+	$+1/3.5\sqrt{2}$
31	21	32	0100−	0
31	21	32	1000−	$-i/3.5\sqrt{2.3}$
31	21	32	1100+	$-i/3.5\sqrt{2}$
31	3	1^2	0000+	$+1/3.5$

31	**31**	**1**		
31	3	2	0000+	$-1/3.5$
31	3	32	0000+	$-1/3.5\sqrt{2.3}$
31	3	32	0100−	$-1/3.5\sqrt{2}$
32	2^2	21	0000+	$+1/2.5\sqrt{2.3}$
32	2^2	21	0100−	$-i/3.5\sqrt{2}$
32	2^2	3	0000+	$+1/3.5\sqrt{2}$
32	2^2	3^2	0000+	$+1/3.5$
32	31	0	0000+	$-1/3.5$
31	**31**	**2^2**		
1^2	1	21	0000+	$+1/4\sqrt{5}$
2	2^2	21	0000+	$+1/4.3\sqrt{3.5}$
2	2^2	3	0000+	$-1/3\sqrt{2.3.5}$
2	2^2	3^2	0000+	$-1/3\sqrt{2.3.5}$
2^2	0	32	0000+	$+1/3\sqrt{2.5}$
21	1^2	2^2	0000+	$+1/4\sqrt{3.5}$
21	1^2	31	0000+	$-1/2.3\sqrt{2.5}$
21	1^2	31	0100−	$+i/3\sqrt{2.3.5}$
21	2	1	0000+	$-\sqrt{3}/4.5$
21	2	2^2	0000+	$-1/2.5\sqrt{2}$
21	2	31	0000+	$+1/4.5$
21	2	31	0100−	$-i/2.3.5\sqrt{3}$
3^2	1^2	2^2	0000+	$+1/2.5\sqrt{3}$
3^2	1^2	31	0000+	$+1/3.5\sqrt{3}$
31	21	1^2	0000+	$-1/2.3.5\sqrt{2}$
31	21	2	0000+	$-7/4.9.5$
31	21	32	0000+	$+1/2.9.5$
31	21	32	0100−	$+1/3.5\sqrt{3}$
31	21	32	1000−	$-2i/9.5\sqrt{3}$
31	21	32	1100+	$-i/9.5$
31	3	1^2	0000+	$-1/3.5$
31	3	2	0000+	$+1/2.9$
31	3	32	0000+	$+2\sqrt{2}/9.5\sqrt{3}$
31	3	32	0100−	$+\sqrt{2}/9.5$
31	3^2	2	0000+	$-1/9.5$
31	3^2	32	0000+	$-1/2.9\sqrt{3}$
31	3^2	32	0100−	$-1/2.9.5$
32	1	21	0000+	$-1/4.5\sqrt{3}$
32	1	21	0100−	$-i/2.3.5$
32	1	3^2	0000+	$+1/3.5$
32	2^2	21	0000+	$-1/2.3.5\sqrt{2}$
32	2^2	21	0100−	$+i\sqrt{2}/3.5\sqrt{3}$
32	2^2	3	0000+	$-\sqrt{2}/3.5\sqrt{3}$
32	2^2	3^2	0000+	$-1/2.3.5\sqrt{3}$
32	31	0	0000+	$+1/3.5$
31	**31**	**31**		
21	2	1	0000+	$+1/2.5\sqrt{3}$
21	2	1	0001−	$+1/3.5$
21	2	2^2	0000+	$-1/3.5\sqrt{2}$
21	2	2^2	0001−	0
3	2	2^2	0000+	$+1/2.3.5$
3	2	2^2	0001−	$+1/2.5\sqrt{3}$
3^2	2	1	0000+	$-1/2.5\sqrt{3}$
3^2	2	1	0001−	$+1/2.3.5$
3^2	2	2^2	0000+	$+1/2.3.5$
3^2	2	2^2	0001−	$-1/2.5\sqrt{3}$

Table 4—*continued*

31	**31**	**31**		
31	21	1^2	0000+	$+1/3.5\sqrt{2}$
31	21	1^2	0001−	0
31	21	1^2	0010−	$+i/3.5\sqrt{2.3}$
31	21	1^2	0011+	$+i/3.5\sqrt{2}$
31	21	2	0000+	$+1/2.9.5$
31	21	2	0001−	$+1/3.5\sqrt{3}$
31	21	2	0010−	$+2i/9.5\sqrt{3}$
31	21	2	0011+	$+i/9.5$
31	3	1^2	0000+	$-1/3.5\sqrt{2.3}$
31	3	1^2	0001−	$-1/3.5\sqrt{2}$
31	3	2	0000+	$+2\sqrt{2}/9.5\sqrt{3}$
31	3	2	0001−	$+\sqrt{2}/9.5$
31	3^2	2	0000+	$-1/2.9\sqrt{3}$
31	3^2	2	0001−	$-1/2.9.5$
32	31	0	0000+	$-1/3.5$
32	31	0	0001−	0
32	31	0	0010−	0
32	31	0	0011+	$+1/3.5$
32	31	21	0000+	$+1/9.5$
32	31	21	0001−	0
32	31	21	0010−	0
32	31	21	0011+	0
32	31	21	0100−	$+i/2.9.5\sqrt{3}$
32	31	21	0101+	$-i/2.3.5$
32	31	21	0110+	$+i/2.3.5$
32	31	21	0111−	$-i/2.3.5\sqrt{3}$
32	31	21	1000−	$-i/2.9.5\sqrt{3}$
32	31	21	1001+	$-i/2.3.5$
32	31	21	1010+	$+i/2.3.5$
32	31	21	1011−	$+i/2.3.5\sqrt{3}$
32	31	21	1100+	$+4/9.3.5$
32	31	21	1101−	0
32	31	21	1110−	0
32	31	21	1111+	$-1/9.5$
32	31	3	0000+	$-7/2.9.3.5$
32	31	3	0001−	$-1/2.3.5\sqrt{3}$
32	31	3	0010−	$-1/2.3.5\sqrt{3}$
32	31	3	0011+	$+1/2.9.5$
32	31	3^2	0000+	$-7/2.9.3.5$
32	31	3^2	0001−	$+1/2.3.5\sqrt{3}$
32	31	3^2	0010−	$+1/2.3.5\sqrt{3}$
32	31	3^2	0011+	$+1/2.9.5$
32	**31**	**0**		
0	0	31	0000+	$+1/\sqrt{3.5}$
1	1	2	0000+	$+1/3\sqrt{5}$
1	1	32	0000+	$-1/3\sqrt{5}$
1^2	1^2	21	0000+	$+1/3\sqrt{5}$
1^2	1^2	3	0000+	$-1/3\sqrt{5}$
2	2	21	0000+	$-1/3\sqrt{2.5}$
2	2	3	0000+	$+1/3\sqrt{2.5}$
2	2	3^2	0000+	$+1/3\sqrt{2.5}$
2^2	2^2	1^2	0000+	$+1/3\sqrt{2.5}$
2^2	2^2	2	0000+	$-1/3\sqrt{2.5}$
2^2	2^2	32	0000+	$+1/3\sqrt{2.5}$
21	21	1	0000+	$+1/2\sqrt{2.3.5}$
32	**31**	**0**		
21	21	2^2	0000+	$-1/2\sqrt{2.3.5}$
21	21	31	0000+	$+1/2\sqrt{2.3.5}$
21	21	31	0100−	0
21	21	31	1000−	0
21	21	31	1100+	$-1/2\sqrt{2.3.5}$
3	3	2^2	0000+	$+1/5\sqrt{2.3}$
3	3	31	0000+	$-1/5\sqrt{2.3}$
3^2	3^2	1	0000+	$-1/5\sqrt{2.3}$
3^2	3^2	2^2	0000+	$+1/5\sqrt{2.3}$
3^2	3^2	31	0000+	$-1/5\sqrt{2.3}$
31	31	1^2	0000+	$-1/3.5$
31	31	2	0000+	$+1/3.5$
31	31	32	0000+	$-1/3.5$
31	31	32	0100−	0
31	31	32	1000−	0
31	31	32	1100+	$+1/3.5$
32	32	0	0000+	$+1/3.5$
32	**31**	**21**		
1	1	2	0000+	$+1/4.3\sqrt{5}$
1	1	2	0001−	$+i/2\sqrt{3.5}$
1	1	32	0000+	$+1/2.3\sqrt{5}$
1	1	32	0001−	0
1^2	1^2	21	0000+	$+\sqrt{5}/8.3$
1^2	1^2	21	0001−	$-i/4\sqrt{3.5}$
1^2	1^2	3	0000+	$+1/2.3\sqrt{5}$
1^2	1^2	3	0001−	$+i/4\sqrt{3.5}$
2	1^2	21	0000+	$-1/8\sqrt{5}$
2	1^2	21	0001−	$-i/4\sqrt{3.5}$
2	1^2	3	0000+	0
2	1^2	3	0001−	$-i/2\sqrt{2.3.5}$
2	2	21	0000+	$-1/8.3.5$
2	2	21	0001−	$-i\sqrt{3}/4.5$
2	2	3	0000+	$+1/2.3.5$
2	2	3	0001−	$+i/2.5\sqrt{3}$
2	2	3^2	0000+	$-1/3.5$
2	2	3^2	0001−	0
2^2	1	2	0000+	$+1/2\sqrt{2.3.5}$
2^2	1	2	0001−	0
2^2	1	32	0000+	$-1/2.3\sqrt{2.5}$
2^2	1	32	0001−	$+i/3\sqrt{2.3.5}$
2^2	2^2	1^2	0000+	$+1/4.3$
2^2	2^2	1^2	0001−	$-i/2.5\sqrt{3}$
2^2	2^2	2	0000+	$-1/2.3.5$
2^2	2^2	2	0001−	$+i/2.5\sqrt{3}$
2^2	2^2	32	0000+	$+1/4.5$
2^2	2^2	32	0001−	$-i/2.3.5\sqrt{3}$
21	0	31	0000+	$+1/2\sqrt{2.3.5}$
21	0	31	0001−	0
21	0	31	0100−	0
21	0	31	0101+	$-1/2\sqrt{2.3.5}$
21	21	1	0000+	$+7/8.5\sqrt{2.3}$
21	21	1	0001−	$+i/4.5\sqrt{2}$
21	21	1	0010−	$-i/8\sqrt{2.3.5}$
21	21	1	0011+	$-1/4\sqrt{2.5}$
21	21	2^2	0000+	$-1/8.5\sqrt{2.3}$
32	**31**	**21**		
21	21	2^2	0001−	$+i/4.5\sqrt{2}$
21	21	2^2	0010−	$+i\sqrt{3}/8\sqrt{2.5}$
21	21	2^2	0011+	$+1/4.3\sqrt{2.5}$
21	21	31	0000+	$-1/8.5\sqrt{2.3}$
21	21	31	0001−	$-i/4.5\sqrt{2}$
21	21	31	0010−	$-i/8\sqrt{2.3.5}$
21	21	31	0011+	$-1/4.3\sqrt{2.5}$
21	21	31	0100−	$-i/4.5\sqrt{2}$
21	21	31	0101+	$-1/4.5\sqrt{2.3}$
21	21	31	0110+	$-1/4.3\sqrt{2.5}$
21	21	31	0111−	$-i/4.3\sqrt{2.3.5}$
21	21	31	1000−	$-i/4.5\sqrt{2}$
21	21	31	1001+	$-1/4.5\sqrt{2.3}$
21	21	31	1010+	$-1/4.3\sqrt{2.5}$
21	21	31	1011−	$-i/4.3\sqrt{2.3.5}$
21	21	31	1100+	$-1/4.5\sqrt{2.3}$
21	21	31	1101−	$-7i/8.3.5\sqrt{2}$
21	21	31	1110−	$-i/4.3\sqrt{2.3.5}$
21	21	31	1111+	$+11/8.9\sqrt{2.5}$
3	21	2^2	0000+	$+1/2.5\sqrt{2.3}$
3	21	2^2	0001−	$+i/3.5\sqrt{2}$
3	21	31	0000+	$+1/2.3.5\sqrt{2}$
3	21	31	0001−	$+i\sqrt{2}/3.5\sqrt{3}$
3	21	31	1000−	$-i/3.5\sqrt{2.3}$
3	21	31	1001+	$-1/2.9.5\sqrt{2}$
3	3	2^2	0000+	0
3	3	2^2	0001−	$+i/3.5$
3	3	31	0000+	$-1/4.3.5\sqrt{3}$
3	3	31	0001−	$-i/2.9$
3^2	21	1	0000+	$+1/2.5\sqrt{2.3}$
3^2	21	1	0001−	0
3^2	21	2^2	0000+	$+1/4.5\sqrt{3}$
3^2	21	2^2	0001−	$-i/2.3.5$
3^2	21	31	0000+	$+1/2.3.5\sqrt{2}$
3^2	21	31	0001−	$-i/3.5\sqrt{2.3}$
3^2	21	31	1000−	$+i\sqrt{2}/3.5\sqrt{3}$
3^2	21	31	1001+	$-1/2.9.5\sqrt{2}$
3^2	3^2	1	0000+	$-1/2.5\sqrt{3}$
3^2	3^2	1	0001−	$+i/4.5$
3^2	3^2	2^2	0000+	$+1/2.5\sqrt{3}$
3^2	3^2	2^2	0001−	$-i/2.3.5$
3^2	3^2	31	0000+	$-1/2.3.5\sqrt{3}$
3^2	3^2	31	0001−	$+7i/4.9.5$
31	1	2	0000+	$-1/4.5\sqrt{3}$
31	1	2	0001−	$+i/2.3.5$
31	1	32	0000+	$+1/3.5\sqrt{2}$
31	1	32	0001−	$+i/3.5\sqrt{2.3}$
31	1	32	0100−	0
31	1	32	0101+	$-i/3.5\sqrt{2}$
31	2^2	1^2	0000+	$+1/2.5\sqrt{2.3}$
31	2^2	1^2	0001−	$+i/3.5\sqrt{2}$
31	2^2	2	0000+	$-1/2.3.5\sqrt{2}$
31	2^2	2	0001−	$-i\sqrt{2}/3.5\sqrt{3}$
31	2^2	32	0000+	$+1/2.9.5$
31	2^2	32	0001−	$+2i/9.5\sqrt{3}$

Table 4—*continued*

32	**31**	**21**		
31	2^2	32	0100−	$-1/3.5\sqrt{3}$
31	2^2	32	0101+	$-i/9.5$
31	31	1^2	0000+	$-1/2.3.5$
31	31	1^2	0001−	$-i/3.5\sqrt{3}$
31	31	1^2	0010−	$+i/3.5\sqrt{3}$
31	31	1^2	0011+	$+2/9.5$
31	31	2	0000+	$+7/4.9.5$
31	31	2	0001−	$+i/2.9\sqrt{3}$
31	31	2	0010−	$-i/2.9\sqrt{3}$
31	31	2	0011+	$-1/9.3$
31	31	32	0000+	$+1/9.5$
31	31	32	0001−	$-i/2.9.5\sqrt{3}$
31	31	32	0010−	$+i/2.9.5\sqrt{3}$
31	31	32	0011+	$+4/9.3.5$
31	31	32	0100−	0
31	31	32	0101+	$+i/2.3.5$
31	31	32	0110+	$+i/2.3.5$
31	31	32	0111−	0
31	31	32	1000−	0
31	31	32	1001+	$-i/2.3.5$
31	31	32	1010+	$-i/2.3.5$
31	31	32	1011−	0
31	31	32	1100+	0
31	31	32	1101−	$+i/2.3.5\sqrt{3}$
31	31	32	1110−	$-i/2.3.5\sqrt{3}$
31	31	32	1111+	$-1/9.5$
32	1^2	21	0000+	$+1/8.5$
32	1^2	21	0001−	$+i/4.5\sqrt{3}$
32	1^2	21	0100−	$-i/4.5\sqrt{3}$
32	1^2	21	0101+	$-1/2.3.5$
32	1^2	3	0000+	$-1/2.5\sqrt{3}$
32	1^2	3	0001−	$+i/4.3.5$
32	2	21	0000+	$-7/8.3.5$
32	2	21	0001−	$+i/4.3.5\sqrt{3}$
32	2	21	0100−	$-i/4.3.5\sqrt{3}$
32	2	21	0101+	$-1/2.9.5$
32	2	3	0000+	$-\sqrt{2}/3.5\sqrt{3}$
32	2	3	0001−	$-i/2.9.5\sqrt{2}$
32	2	3^2	0000+	$-1/3.5\sqrt{3}$
32	**31**	**21**		
32	2	3^2	0001−	$+2i/9.5$
32	32	0	0000+	$+1/3.5$
32	32	0	0001−	0
32	32	0	0010−	0
32	32	0	0011+	$-1/3.5$
32	32	21	0000+	$+1/8.5$
32	32	21	0001−	$-i/4.3.5\sqrt{3}$
32	32	21	0010−	$-i/4.3.5\sqrt{3}$
32	32	21	0011+	$-2/9.5$
32	32	21	0100−	$+i/4.3.5\sqrt{3}$
32	32	21	0101+	$-1/9.5$
32	32	21	0110+	$-1/9.5$
32	32	21	0111−	$-7i/4.9.5\sqrt{3}$
32	32	21	1000−	$+i/4.3.5\sqrt{3}$
32	32	21	1001+	$-1/9.5$
32	32	21	1010+	$-1/9.5$
32	32	21	1011−	$-7i/4.9.5\sqrt{3}$
32	32	21	1100+	$-2/9.5$
32	32	21	1101−	$+7i/4.9.5\sqrt{3}$
32	32	21	1110−	$+7i/4.9.5\sqrt{3}$
32	32	21	1111+	$+5/8.9.3$
32	**31**	**3**		
0	3^2	31	0000+	$-1/5\sqrt{2.3}$
1	2^2	2	0000+	$+1/2.5$
1	2^2	32	0000+	$+1/3.5\sqrt{3}$
1^2	32	21	0000+	$-1/2.5\sqrt{3}$
1^2	32	21	1000−	$-i/4.3.5$
1^2	32	3	0000+	$+1/3.5$
2	1^2	21	0000+	$-1/5\sqrt{2.3}$
2	1^2	3	0000+	$+1/2.5\sqrt{3}$
2	32	21	0000+	$-\sqrt{2}/3.5\sqrt{3}$
2	32	21	1000−	$+i/2.9.5\sqrt{2}$
2	32	3	0000+	$-1/2.9$
2	32	3^2	0000+	$+\sqrt{2}/9.5$
2^2	31	1^2	0000+	$+1/3.5\sqrt{2}$
2^2	31	2	0000+	$-\sqrt{2}/3.5\sqrt{3}$
2^2	31	32	0000+	$-1/2.9\sqrt{3}$
2^2	31	32	1000−	$-1/2.9.5$
21	21	1	0000+	$-1/2.5\sqrt{2}$
32	**31**	**3**		
21	21	2^2	0000+	$-1/2.3.5\sqrt{2}$
21	21	31	0000+	$+1/2.3.5\sqrt{2}$
21	21	31	0100−	$+i\sqrt{2}/3.5\sqrt{3}$
21	21	31	1000−	$-i/3.5\sqrt{2.3}$
21	21	31	1100+	$-1/2.9.5\sqrt{2}$
21	3^2	1	0000+	$-1/4.5$
21	3^2	2^2	0000+	$+1/2.3.5\sqrt{2}$
21	3^2	31	0000+	$-1/4.3.5\sqrt{3}$
21	3^2	31	0100−	$-i/2.9$
3	0	31	0000+	$-1/5\sqrt{2.3}$
3	21	2^2	0000+	$-1/3.5$
3	21	31	0000+	$-1/2.3.5\sqrt{3}$
3	21	31	1000−	$+7i/4.9.5$
3^2	3	2^2	0000+	$+1/3.5\sqrt{2}$
3^2	3	31	0000+	$-2/9.5$
31	1	2	0000+	$+1/3.5$
31	1	32	0000+	$-1/3.5\sqrt{2.3}$
31	1	32	0100−	$+1/3.5\sqrt{2}$
31	2^2	1^2	0000+	$+1/3.5$
31	2^2	2	0000+	$-1/2.3.5\sqrt{3}$
31	2^2	32	0000+	$+2\sqrt{2}/9.5\sqrt{3}$
31	2^2	32	0100−	$-\sqrt{2}/9.5$
31	31	1^2	0000+	$+2/9.5$
31	31	2	0000+	$-1/9.3$
31	31	32	0000+	$-7/2.9.3.5$
31	31	32	0100−	$+1/2.3.5\sqrt{3}$
31	31	32	1000−	$+1/2.3.5\sqrt{3}$
31	31	32	1100+	$+1/2.9.5$
32	2	21	0000+	$-1/3.5\sqrt{3}$
32	2	21	0100−	$-2i/9.5$
32	2	3	0000+	$+\sqrt{2}/9.5$
32	2	3^2	0000+	$+1/9.5$
32	32	0	0000+	$-1/3.5$
32	32	21	0000+	$-1/2.9.5$
32	32	21	0100−	$-i/9.5\sqrt{3}$
32	32	21	1000−	$-13i/4.9.5\sqrt{3}$
32	32	21	1100+	$+1/2.9.3.5$
32	32	3	0000+	$-1/9.3.5$
32	32	3^2	0000+	$-1/9.3.5$

Table 5. $3j$ symbols of SU_6.

0	0	0	0+	1^3	1^2	1	0+	21^3	21	2^5	0+	31^4	$2^4 1$	2	0+
1^5	1	0	0−	1^3	1^3	0	0−	21^3	$2^4 1$	1^4	0−	31^4	21^3	21^4	0+
1^2	1^5	1^5	0−	1^3	1^3	21^4	0+	21^3	21^3	1^2	0−	31^4	$2^2 1^3$	1^4	0−
1^2	1^2	1^2	0+	21	1^4	1^5	0+	21^3	21^3	2	0+	31^4	$2^2 1^3$	2^5	0+
1^4	1^2	0	0+	21	2^5	1^5	0−	$2^2 1^3$	21^3	0	0−	31^4	3^5	1^2	0−
2	1^5	1^5	0+	21	1^3	21^4	0+	$2^2 1^3$	21^3	21^4	0+	31^4	3^5	2	0+
2^5	2	0	0+	$2^4 1$	21	0	0−	$2^2 1^3$	21^3	21^4	1−	31^4	31^4	1^4	0−
21^4	1^5	1	0+	$2^4 1$	21	21^4	0+	3	2^5	1^5	0−	31^4	31^4	2^5	0+
21^4	1^4	1^2	0−	$2^4 1$	21	21^4	1−	3	$2^4 1$	21^4	0+	32^4	31^4	0	0−
21^4	2	1^4	0+	21^3	1^2	1^5	0+	3^5	3	0	0−	32^4	31^4	21^4	0+
21^4	2^5	2	0−	21^3	21^4	1	0−	3^5	3	21^4	0+	32^4	31^4	21^4	1−
21^4	21^4	0	0+	21^3	1^3	1^4	0−	31^4	2^5	1	0−				
21^4	21^4	21^4	0−	21^3	1^3	2^5	0+	31^4	21^4	1^5	0−				
21^4	21^4	21^4	1+	21^3	21	1^4	0−	31^4	$2^4 1$	1^2	0−				

Table 6. $6j$ symbols of SU_6.

0	**0**	**0**		
0	0	0	0000+	$+1$
1^5	**1**	**0**		
0	0	1	0000+	$-1/\sqrt{2.3}$
1^5	1^5	0	0000+	$-1/2.3$
1^2	**1^5**	**1^5**		
0	1	1^5	0000+	$-1/2.3$
1^2	**1^2**	**1^2**		
1^4	1^2	0	0000+	$+1/3.5$
1^4	**1^2**	**0**		
0	0	1^2	0000+	$+1/\sqrt{3.5}$
1^5	1^5	1	0000+	$-1/3\sqrt{2.5}$
1^2	1^2	1^4	0000+	$+1/3.5$
1^4	1^4	0	0000+	$+1/3.5$
2	**1^5**	**1^5**		
0	1	1^5	0000+	$+1/2.3$
2^5	**2**	**0**		
0	0	2	0000+	$+1/\sqrt{3.7}$
1^5	1^5	1	0000+	$+1/3\sqrt{2.7}$
2^5	2^5	0	0000+	$+1/3.7$
21^4	**1^5**	**1**		
0	1^5	1^5	0000+	$+1/2.3$
1^2	1	1	0000+	$+1/2.3.5$
2	1	1	0000+	$+1/2.3.7$
21^4	1^5	1^5	0000+	$+1/2.3.5.7$
21^4	**1^4**	**1^2**		
0	1^4	1^4	0000+	$-1/3.5$
1	1^5	1^5	0000+	$+1/3.5$
1^4	1^2	1^2	0000+	$-1/2.3.5$
21^4	1^4	1^4	0000+	$+1/2.3.7$
21^4	**2**	**1^4**		
0	1^2	2	0000+	$+1/3\sqrt{5.7}$
1^5	1	1	0000+	$+1/\sqrt{2.3.5.7}$
21^4	1^2	1^2	0000+	$-1/5.7$
21^4	1^2	2	0000+	$+2\sqrt{2}/3.5.7$
21^4	2	1^2	0000+	$+1/5.7$
21^4	**2^5**	**2**		
0	2^5	2^5	0000+	$-1/3.7$
1	1^5	1^5	0000+	$+\sqrt{2}/3.7$
21^4	1^4	1^4	0000+	$+2\sqrt{2}/3.5.7$
21^4	1^4	2^5	0000+	$-1/5.7$
21^4	2^5	1^4	0000+	$-1/5.7$
21^4	2^5	2^5	0000+	$+11/4.3.5.7$
21^4	**21^4**	**0**		
0	0	21^4	0000+	$+1/\sqrt{5.7}$
1	1	1	0000+	$+1/\sqrt{2.3.5.7}$
1^5	1^5	1^5	0000+	$+1/\sqrt{2.3.5.7}$
1^2	1^2	1^2	0000+	$-1/5\sqrt{3.7}$
1^2	1^2	2	0000+	$+1/5\sqrt{3.7}$
1^4	1^4	1^4	0000+	$-1/5\sqrt{3.7}$
1^4	1^4	2^5	0000+	$+1/5\sqrt{3.7}$
2	2	1^2	0000+	$+1/7\sqrt{3.5}$
2	2	2	0000+	$-1/7\sqrt{3.5}$
2^5	2^5	1^4	0000+	$+1/7\sqrt{3.5}$
2^5	2^5	2^5	0000+	$-1/7\sqrt{3.5}$
21^4	21^4	0	0000+	$+1/5.7$
21^4	**21^4**	**21^4**		
1	1	1	0000+	$+\sqrt{3}/5.7$
1	1	1	0001−	$+2i\sqrt{2}/5.7\sqrt{3}$
1^5	1^5	1^5	0000+	$+\sqrt{3}/5.7$

Table 6—*continued*

21^4	**21^4**	**21^4**		
1^5	1^5	1^5	0001−	$-2i\sqrt{2}/5.7\sqrt{3}$
1^2	1^2	1^2	0000+	$-\sqrt{3}/2.5.7$
1^2	1^2	1^2	0001−	$-i/5.7\sqrt{2.3}$
1^4	1^4	1^4	0000+	$-\sqrt{3}/2.5.7$
1^4	1^4	1^4	0001−	$+i/5.7\sqrt{2.3}$
2	1^2	1^2	0000+	$+1/5.7\sqrt{3}$
2	1^2	1^2	0001−	$+i\sqrt{2}/5.7\sqrt{3}$
2	2	1^2	0000+	$-\sqrt{2}/5.7\sqrt{3}$
2	2	1^2	0001−	$-i/5.7\sqrt{3}$
2	2	2	0000+	$+\sqrt{3}/2.5.7\sqrt{2}$
2	2	2	0001−	$+i/4.7\sqrt{3}$
2^5	1^4	1^4	0000+	$+1/5.7\sqrt{3}$
2^5	1^4	1^4	0001−	$-i\sqrt{2}/5.7\sqrt{3}$
2^5	2^5	1^4	0000+	$-\sqrt{2}/5.7\sqrt{3}$
2^5	2^5	1^4	0001−	$+i/5.7\sqrt{3}$
2^5	2^5	2^5	0000+	$+\sqrt{3}/2.5.7\sqrt{2}$
2^5	2^5	2^5	0001−	$-i/4.7\sqrt{3}$
21^4	21^4	0	0000+	$-1/5.7$
21^4	21^4	0	0001−	0
21^4	21^4	0	0010−	0
21^4	21^4	0	0011+	$+1/5.7$
21^4	21^4	21^4	0000+	$+1/2.5.7$
21^4	21^4	21^4	0001−	0
21^4	21^4	21^4	1000−	0
21^4	21^4	21^4	1001+	$-1/2.5.7$
21^4	21^4	21^4	1100+	$-1/2.5.7$
21^4	21^4	21^4	1101−	0
21^4	21^4	21^4	1110−	0
21^4	21^4	21^4	1111+	$+3/8.5.7$
1^3	**1^2**	**1**		
0	1^5	1^2	0000+	$+1/3\sqrt{2.5}$
1^5	1^4	1	0000+	$+1/3.5$
1^2	1	1^4	0000+	$+1/2.5\sqrt{3}$
21^4	1^5	1^2	0000+	$-1/2.3.5$
1^3	1^2	1^5	0000+	$-1/4.5$
1^3	**1^3**	**0**		
0	0	1^3	0000+	$-1/2\sqrt{5}$
1	1	1^4	0000+	$+1/2\sqrt{2.3.5}$
1^5	1^5	1^2	0000+	$+1/2\sqrt{2.3.5}$
1^2	1^2	1^5	0000+	$+1/2.5\sqrt{3}$
1^4	1^4	1	0000+	$+1/2.5\sqrt{3}$
21^4	21^4	1^3	0000+	$+1/2.5\sqrt{7}$
1^3	1^3	0	0000+	$-1/4.5$
1^3	**1^3**	**21^4**		
1	1	1^4	0000+	$+1/2.5\sqrt{2.3}$
1^5	1^5	1^2	0000+	$+1/2.5\sqrt{2.3}$
1^2	1^2	1^5	0000+	$+1/2.5\sqrt{2.3}$
1^4	1^4	1	0000+	$+1/2.5\sqrt{2.3}$
21^4	0	1^3	0000+	$+1/2.5\sqrt{7}$
21^4	21^4	1^3	0000+	$-1/5.7\sqrt{2}$
21^4	21^4	1^3	0010−	0
1^3	1^3	0	0000+	$+1/4.5$
1^3	1^3	21^4	0000+	$-3/4.5.7$
21	**1^4**	**1^5**		
0	1	1^4	0000+	$+1/3\sqrt{2.5}$
21	**1^4**	**1^5**		
1	1^2	1^5	0000+	$-1/2.3.5$
21^4	1	1^4	0000+	$+1/3.5.7$
21	**2^5**	**1^5**		
0	1	2^5	0000+	$-1/3\sqrt{2.7}$
1	1^2	1^5	0000+	$+1/2\sqrt{3.5.7}$
1	2	1^5	0000+	$+1/2.3.7$
21^4	1	1^4	0000+	$-1/5.7\sqrt{2}$
21^4	1	2^5	0000+	$-\sqrt{2}/3.5.7$
21	**1^3**	**21^4**		
0	21^4	1^3	0000+	$+1/2.5\sqrt{7}$
1	1	1^4	0000+	$+1/5\sqrt{3.7}$
1^2	1^2	1^5	0000+	$-1/2.5\sqrt{3.7}$
1^2	2	1^5	0000+	$-1/5\sqrt{2.3.7}$
21^4	21^4	1^3	0000+	$-1/2.5.7\sqrt{2}$
21^4	21^4	1^3	0010−	$-i/2.5.7$
1^3	1^3	21^4	0000+	$+1/2.5.7$
21	1^3	21^4	0000+	$-1/4.5.7$
2^41	1^3	21^4	0000+	$+1/4.5.7$
2^41	**21**	**0**		
0	0	21	0000+	$-1/\sqrt{2.5.7}$
1^5	1^5	1^2	0000+	$+1/2\sqrt{3.5.7}$
1^5	1^5	2	0000+	$-1/2\sqrt{3.5.7}$
1^4	1^4	1	0000+	$+1/5\sqrt{2.3.7}$
2^5	2^5	1	0000+	$-1/7\sqrt{2.3.5}$
21^4	21^4	1^3	0000+	$+1/5.7\sqrt{2}$
21^4	21^4	21	0000+	$+1/5.7\sqrt{2}$
21^4	21^4	21	0100−	0
21^4	21^4	21	1000−	0
21^4	21^4	21	1100+	$-1/5.7\sqrt{2}$
1^3	1^3	21^4	0000+	$+1/2.5\sqrt{2.7}$
2^41	2^41	0	0000+	$-1/2.5.7$
2^41	**21**	**21^4**		
1^5	1^5	1^2	0000+	$+3/2.5.7$
1^5	1^5	1^2	0001−	0
1^5	1^5	2	0000+	$+1/2.9.5.7$
1^5	1^5	2	0001−	$+8i\sqrt{2}/9.5.7$
1^4	1^4	1	0000+	$+1/3.5.7$
1^4	1^4	1	0001−	$-2i\sqrt{2}/3.5.7$
2^5	1^4	1	0000+	$+1/9.5\sqrt{2}$
2^5	1^4	1	0001−	$+4i/9.5.7$
2^5	2^5	1	0000+	$+\sqrt{2}/9.7$
2^5	2^5	1	0001−	$-2i/9.5.7$
21^4	0	21	0000+	$+1/5.7\sqrt{2}$
21^4	0	21	0001−	0
21^4	0	21	0100−	0
21^4	0	21	0101+	$-1/5.7\sqrt{2}$
21^4	21^4	1^3	0000+	$-1/2.3.7\sqrt{3}$
21^4	21^4	1^3	0001−	$+i/3.5.7\sqrt{2.3}$
21^4	21^4	1^3	0010−	$+i/5.7\sqrt{2.3}$
21^4	21^4	1^3	0011+	$-1/4.5.7\sqrt{3}$
21^4	21^4	21	0000+	$-61/9.9.5.7\sqrt{3}$
21^4	21^4	21	0001−	$-4i\sqrt{2}/9.9.5.7\sqrt{3}$
21^4	21^4	21	0010−	$+2i\sqrt{2}/9.3.5\sqrt{3}$
21^4	21^4	21	0011+	$+2/9.3.5.7\sqrt{3}$
21^4	21^4	21	0100−	$-4i\sqrt{2}/9.9.5.7\sqrt{3}$

Table 6—*continued*

$\mathbf{2^41}$	$\mathbf{21}$	$\mathbf{21^4}$		
21^4	21^4	21	0101+	$+13/2.9.9.5.7\sqrt{3}$
21^4	21^4	21	0110+	$+2/9.3.5.7\sqrt{3}$
21^4	21^4	21	0111−	$+11\,i/2.9.3.5.7\sqrt{2.3}$
21^4	21^4	21	1000−	$-4\,i\sqrt{2}/9.9.5.7\sqrt{3}$
21^4	21^4	21	1001+	$+13/2.9.9.5.7\sqrt{3}$
21^4	21^4	21	1010+	$+2/9.3.5.7\sqrt{3}$
21^4	21^4	21	1011−	$+11\,i/2.9.3.5.7\sqrt{2.3}$
21^4	21^4	21	1100+	$+13/2.9.9.5.7\sqrt{3}$
21^4	21^4	21	1101−	$-1289\,i/8.2.9.9.5.7\sqrt{2.3}$
21^4	21^4	21	1110−	$+11\,i/2.9.3.5.7\sqrt{2.3}$
21^4	21^4	21	1111+	$-127/8.8.9.3.5\sqrt{3}$
1^3	1^3	21^4	0000+	$-1/2.5.7\sqrt{2.3}$
1^3	1^3	21^4	0001−	$+\,i/5.7\sqrt{3}$
2^41	1^3	21^4	0000+	$+1/9.5.7$
2^41	1^3	21^4	0001−	$-2\,i\sqrt{2}/9.5.7$
2^41	1^3	21^4	0100−	$+2\,i\sqrt{2}/9.5.7$
2^41	1^3	21^4	0101+	$-17/8.9.5.7$
2^41	2^41	0	0000+	$+1/2.5.7$
2^41	2^41	0	0001−	0
2^41	2^41	0	0010−	0
2^41	2^41	0	0011+	$-1/2.5.7$
2^41	2^41	21^4	0000+	$-1/2.9.9.7$
2^41	2^41	21^4	0001−	$+8\,i\sqrt{2}/9.9.5.7$
2^41	2^41	21^4	0010−	$+8\,i\sqrt{2}/9.9.5.7$
2^41	2^41	21^4	0011+	$+61/2.9.9.5.7$
2^41	2^41	21^4	0100−	$-8\,i\sqrt{2}/9.9.5.7$
2^41	2^41	21^4	0101+	$+2/9.9.7$
2^41	2^41	21^4	0110+	$+2/9.9.7$
2^41	2^41	21^4	0111−	$+13\,i/8.9.9.5\sqrt{2}$
2^41	2^41	21^4	1000−	$-8\,i\sqrt{2}/9.9.5.7$
2^41	2^41	21^4	1001+	$+2/9.9.7$
2^41	2^41	21^4	1010+	$+2/9.9.7$
2^41	2^41	21^4	1011−	$+13\,i/8.9.9.5\sqrt{2}$
2^41	2^41	21^4	1100+	$+61/2.9.9.5.7$
2^41	2^41	21^4	1101−	$-13\,i/8.9.9.5\sqrt{2}$
2^41	2^41	21^4	1110−	$-13\,i/8.9.9.5\sqrt{2}$
2^41	2^41	21^4	1111+	$-67/8.8.9.9.5.7$
$\mathbf{21^3}$	$\mathbf{1^2}$	$\mathbf{1^5}$		
0	1	1^2	0000+	$+1/3\sqrt{2.5}$
21^4	1	1^2	0000+	$+1/2.3.5.7$
1^3	1^4	1^5	0000+	$-1/4.3.5$
2^41	1^4	1^5	0000+	$-1/3.5.7$
$\mathbf{21^3}$	$\mathbf{21^4}$	$\mathbf{1}$		
0	1^5	21^4	0000+	$-1/\sqrt{2.3.5.7}$
1	21^4	1	0000+	$+1/5.7$
1^5	1^4	1^5	0000+	$+1/5\sqrt{3.7}$
1^2	1	1^2	0000+	$+1/2.5\sqrt{3.7}$
2	1	1^2	0000+	$-1/3.7\sqrt{2.5}$
21^4	1^5	21^4	0000+	$-1/2.5.7\sqrt{3}$
21^4	1^5	21^4	0100−	$+\,i/5.7\sqrt{2.3}$
21^3	1^4	1^5	0000+	$-1/4.3.5.7$
2^21^3	21^4	1	0000+	$-1/4.5.7$
$\mathbf{21^3}$	$\mathbf{1^3}$	$\mathbf{1^4}$		
0	1^2	1^3	0000+	$-1/2.5\sqrt{3}$
1	1^3	1^4	0000+	$+1/2.3.5$
$\mathbf{21^3}$	$\mathbf{1^3}$	$\mathbf{1^4}$		
1^5	1	1^2	0000+	$+1/2.5\sqrt{2.3}$
1^2	1^4	1^5	0000+	$+1/2.3.5\sqrt{2}$
1^4	21^4	1	0000+	$+1/2.5\sqrt{2.7}$
21^4	1^2	1^3	0000+	$-1/5.7\sqrt{2.3}$
21^4	2	1^3	0000+	$+1/2.7\sqrt{3.5}$
1^3	1^5	21^4	0000+	$-1/2.5\sqrt{3.7}$
2^41	1^5	21^4	0000+	$+1/2.5.7\sqrt{2.3}$
21^3	1	1^2	0000+	$-1/2.3.5.7$
2^21^3	1^3	1^4	0000+	$-1/4.3.5.7$
$\mathbf{21^3}$	$\mathbf{1^3}$	$\mathbf{2^5}$		
0	2	1^3	0000+	$+1/2\sqrt{3.5.7}$
1^5	1	1^2	0000+	$+1/2.3\sqrt{2.7}$
1^4	21^4	1	0000+	$-1/2.3\sqrt{5.7}$
21^4	1^2	1^3	0000+	$+1/2.7\sqrt{3.5}$
21^4	2	1^3	0000+	$-1/5.7\sqrt{3}$
2^41	1^5	21^4	0000+	$-1/2.7\sqrt{2.3.5}$
2^21^3	1^3	1^4	0000+	$+1/4.3.7$
2^21^3	1^3	2^5	0000+	$+1/4.5.7$
$\mathbf{21^3}$	$\mathbf{21}$	$\mathbf{1^4}$		
0	1^2	21	0000+	$-1/5\sqrt{2.3.7}$
1^5	1	1^2	0000+	$-1/5\sqrt{3.7}$
1^4	21^4	1	0000+	$+1/2.5.7$
2^5	21^4	1	0000+	$+1/5.7\sqrt{2}$
21^4	1^2	1^3	0000+	$+\sqrt{3}/2.5.7\sqrt{2}$
21^4	1^2	21	0000+	$-11/2.9.5.7$
21^4	1^2	21	0100−	$-\,i/2.9.5.7\sqrt{2}$
21^4	2	1^3	0000+	$-1/2.7\sqrt{3.5}$
21^4	2	21	0000+	$-4\sqrt{2}/9.5.7$
21^4	2	21	0100−	$+\,i/4.9.5.7$
1^3	1^5	21^4	0000+	$-1/4.5\sqrt{3.7}$
2^41	1^5	21^4	0000+	$+1/9.5.7$
2^41	1^5	21^4	0100−	$-\,i/9.7\sqrt{2}$
21^3	1	1^2	0000+	$+1/4.3.5.7$
2^21^3	1^3	1^4	0000+	$-1/2.3.7\sqrt{5}$
2^21^3	21	1^4	0000+	$-1/4.3.7$
2^21^3	2^41	1^4	0000+	$+1/4.3.5.7$
$\mathbf{21^3}$	$\mathbf{21}$	$\mathbf{2^5}$		
0	2	21	0000+	$+1/7\sqrt{2.3.5}$
1^5	1	1^2	0000+	$+2/3.7\sqrt{5}$
1^4	21^4	1	0000+	$+\sqrt{2}/3.5.7$
2^5	21^4	1	0000+	$+1/5.7\sqrt{2}$
21^4	1^2	1^3	0000+	$+1/5.7\sqrt{3}$
21^4	1^2	21	0000+	$-4\sqrt{2}/9.5.7$
21^4	1^2	21	0100−	$+\,i/4.9.5.7$
21^4	2	1^3	0000+	$-1/2.7\sqrt{3.5}$
21^4	2	21	0000+	$-1/2.3.7\sqrt{2}$
21^4	2	21	0100−	$-\,i/8.2.3.5.7$
2^41	1^5	21^4	0000+	$+2/9.5.7$
2^41	1^5	21^4	0100−	$+\,i/2.9.5\sqrt{2}$
2^21^3	1^3	1^4	0000+	$-1/3.5.7$
2^21^3	21	1^4	0000+	$+1/4.3.7$
$\mathbf{21^3}$	$\mathbf{2^41}$	$\mathbf{1^4}$		
0	1^2	2^41	0000+	$-1/5\sqrt{2.3.7}$
1	1^3	1^4	0000+	$+1/2.3\sqrt{2.5.7}$
1	1^3	2^5	0000+	$+1/2.5\sqrt{2.7}$

Table 6—*continued*

$\mathbf{21^3}$	$\mathbf{2^41}$	$\mathbf{1^4}$		
1	21	1^4	0000+	$-1/3.5.7$
1^2	1^4	1^5	0000+	$-1/2.3\sqrt{5.7}$
21^4	1^2	1^3	0000+	$-1/2.7\sqrt{2.3.5}$
21^4	1^2	2^41	0000+	$-1/2.3.5.7$
21^4	1^2	2^41	0100−	$-i\sqrt{2}/3.5.7$
21^4	2	1^3	0000+	$-1/2.5.7\sqrt{3}$
1^3	1^5	21^4	0000+	$+1/4\sqrt{3.5.7}$
21^3	1	1^2	0000+	$-1/4.3.7$
2^21^3	1^3	1^4	0000+	$-1/2.3.7\sqrt{5}$
2^21^3	1^3	2^5	0000+	$-1/3.5.7$
2^21^3	21	1^4	0000+	$+1/4.3.5.7$
2^21^3	2^41	1^4	0000+	$-1/4.3.7$
$\mathbf{21^3}$	$\mathbf{21^3}$	$\mathbf{1^2}$		
1	1^5	21^4	0000+	$-1/2.5\sqrt{2.7}$
1^2	0	2^21^3	0000+	$-1/2.3\sqrt{5.7}$
1^4	1^2	1^3	0000+	$-\sqrt{2}/3.5\sqrt{3.7}$
1^4	1^2	21	0000+	$+1/2.3\sqrt{2.3.5.7}$
1^4	1^2	2^41	0000+	$+1/2.3\sqrt{2.3.5.7}$
21^4	1^4	1^5	0000+	$+\sqrt{3}/2.5.7\sqrt{2}$
21^4	1^4	21^3	0000+	$+1/4.3.5\sqrt{2.3}$
21^4	1^4	21^3	0100−	$-i/4.3.7\sqrt{2}$
21^4	2^5	21^3	0000+	$+1/3.7\sqrt{3.5}$
21^4	2^5	21^3	0100−	0
1^3	1	1^2	0000+	$+1/2.5\sqrt{3.7}$
21	1	1^2	0000+	$+1/2.5.7\sqrt{2.3}$
21^3	1^3	1^4	0000+	$-11/4.9.5.7$
21^3	1^3	2^5	0000+	$-1/4.3.7$
21^3	21	1^4	0000+	$+1/2.9.7$
21^3	2^41	1^4	0000+	$+2/9.5.7$
21^3	2^41	2^5	0000+	$-1/4.3.7\sqrt{5}$
2^21^3	1^5	21^4	0000+	$+1/3.5.7$
2^21^3	1^5	21^4	0100−	0
2^21^3	21^3	0	0000+	$-1/4.3.7$
$\mathbf{21^3}$	$\mathbf{21^3}$	$\mathbf{2}$		
1	1^5	21^4	0000+	$+1/2.7\sqrt{2.3}$
2	0	2^21^3	0000+	$+1/2.3.7$
21^4	1^4	1^5	0000+	$+1/2.7\sqrt{3.5}$
21^4	1^4	21^3	0000+	$+1/3.7\sqrt{3.5}$
21^4	1^4	21^3	0100−	0
21^4	2^5	21^3	0000+	$+1/8.7\sqrt{3}$
21^4	2^5	21^3	0100−	$-i/8.3.7$
21	1	1^2	0000+	$+1/2.7\sqrt{2.3.5}$
21^3	1^3	1^4	0000+	$-1/4.3.7$
21^3	1^3	2^5	0000+	$-1/4.3.7$
21^3	2^41	1^4	0000+	$-1/4.3.7\sqrt{5}$
21^3	2^41	2^5	0000+	$+1/8.3.7$
2^21^3	21^3	0	0000+	$+1/4.3.7$
$\mathbf{2^21^3}$	$\mathbf{21^3}$	$\mathbf{0}$		
0	0	21^3	0000+	$-1/2\sqrt{3.7}$
1	1	21^4	0000+	$-1/2.3\sqrt{2.7}$
1^5	1^5	1^4	0000+	$+1/2.3\sqrt{2.7}$
1^2	1^2	1	0000+	$+1/2.3\sqrt{5.7}$
1^2	1^2	2^21^3	0000+	$-1/2.3\sqrt{5.7}$
1^4	1^4	1^3	0000+	$-1/2.3\sqrt{5.7}$
1^4	1^4	21	0000+	$-1/2.3\sqrt{5.7}$
1^4	1^4	2^41	0000+	$-1/2.3\sqrt{5.7}$
2	2	2^21^3	0000+	$+1/2.3.7$
2^5	2^5	1^3	0000+	$+1/2.3.7$
2^5	2^5	2^41	0000+	$+1/2.3.7$
21^4	21^4	1^5	0000+	$-1/2.7\sqrt{3.5}$
21^4	21^4	21^3	0000+	$+1/2.7\sqrt{3.5}$
21^4	21^4	21^3	0100−	0
21^4	21^4	21^3	1000−	0
21^4	21^4	21^3	1100+	$-1/2.7\sqrt{3.5}$
1^3	1^3	1^2	0000+	$-1/4\sqrt{3.5.7}$
1^3	1^3	2	0000+	$+1/4\sqrt{3.5.7}$
21	21	1^2	0000+	$-1/2.7\sqrt{2.3.5}$
21	21	2	0000+	$+1/2.7\sqrt{2.3.5}$
2^41	2^41	1^2	0000+	$-1/2.7\sqrt{2.3.5}$
21^3	21^3	1^4	0000+	$-1/4.3.7$
21^3	21^3	2^5	0000+	$+1/4.3.7$
2^21^3	2^21^3	0	0000+	$-1/4.3.7$
$\mathbf{2^21^3}$	$\mathbf{21^3}$	$\mathbf{21^4}$		
1	1	21^4	0000+	$+3\sqrt{3}/4.5.7\sqrt{2}$
1	1	21^4	0001−	$-i/4.3.7\sqrt{2}$
1^5	1^5	1^4	0000+	$+\sqrt{3}/4.5.7\sqrt{2}$
1^5	1^5	1^4	0001−	$+5i/4.3.7\sqrt{2}$
1^2	1^2	1	0000+	$-3\sqrt{3}/4.5.7\sqrt{2}$
1^2	1^2	1	0001−	$+i/4.3.7\sqrt{2}$
1^2	1^2	2^21^3	0000+	$+1/4.3.5\sqrt{2.3}$
1^2	1^2	2^21^3	0001−	$-i/4.3.7\sqrt{2}$
1^4	1^4	1^3	0000+	$+1/2.5.7\sqrt{2.3}$
1^4	1^4	1^3	0001−	$-i/2.3.7\sqrt{2}$
1^4	1^4	21	0000+	$+1/4.7\sqrt{2.3}$
1^4	1^4	21	0001−	$-i/4.3.5.7\sqrt{2}$
1^4	1^4	2^41	0000+	$-1/2.5.7\sqrt{2.3}$
1^4	1^4	2^41	0001−	$-i/2.3.7\sqrt{2}$
2	1^2	2^21^3	0000+	$+1/3.7\sqrt{3.5}$
2	1^2	2^21^3	0001−	0
2	2	2^21^3	0000+	$+1/8.7\sqrt{3}$
2	2	2^21^3	0001−	$-i/8.3.7$
2^5	1^4	1^3	0000+	$+1/4.7\sqrt{3.5}$
2^5	1^4	1^3	0001−	$+i/4.7\sqrt{5}$
2^5	1^4	2^41	0000+	$-1/8.5.7\sqrt{3}$
2^5	1^4	2^41	0001−	$-i/8.7$
2^5	2^5	1^3	0000+	0
2^5	2^5	1^3	0001−	$-2i/3.5.7$
2^5	2^5	2^41	0000+	$-3\sqrt{3}/8.4.5.7$
2^5	2^5	2^41	0001−	$-11i/8.4.3.7$
21^4	0	21^3	0000+	$+1/2.7\sqrt{3.5}$
21^4	0	21^3	0001−	0
21^4	0	21^3	0100−	0
21^4	0	21^3	0101+	$-1/2.7\sqrt{3.5}$
21^4	21^4	1^5	0000+	$-1/4.5.7\sqrt{2}$
21^4	21^4	1^5	0001−	$+i/4.7\sqrt{2.3}$
21^4	21^4	1^5	0010−	$-3i/8.5.7$
21^4	21^4	1^5	0011+	$+1/8.7\sqrt{3}$
21^4	21^4	21^3	0000+	$+523/8.8.2.9.5.7\sqrt{2}$
21^4	21^4	21^3	0001−	$-i/8.8.2\sqrt{2.3}$
21^4	21^4	21^3	0010−	$+181i/8.8.8.3.5.7$

Table 6—*continued*

2^21^3	**21^3**	**21^4**		
21^4	21^4	21^3	0011+	$+11/8.8.8.7\sqrt{3}$
21^4	21^4	21^3	0100−	$-i/8.8.2\sqrt{2.3}$
21^4	21^4	21^3	0101+	$+3/8.8.2.7\sqrt{2}$
21^4	21^4	21^3	0110+	$+11/8.8.8.7\sqrt{3}$
21^4	21^4	21^3	0111−	$-9i/8.8.8.7$
21^4	21^4	21^3	1000−	$-i/8.8.2\sqrt{2.3}$
21^4	21^4	21^3	1001+	$+3/8.8.2.7\sqrt{2}$
21^4	21^4	21^3	1010+	$+11/8.8.8.7\sqrt{3}$
21^4	21^4	21^3	1011−	$-9i/8.8.8.7$
21^4	21^4	21^3	1100+	$+3/8.8.2.7\sqrt{2}$
21^4	21^4	21^3	1101−	$+43i\sqrt{3}/8.8.2.5.7\sqrt{2}$
21^4	21^4	21^3	1110−	$-9i/8.8.8.7$
21^4	21^4	21^3	1111+	$+71/8.8.8.7\sqrt{3}$
1^3	1^3	1^2	0000+	$-11/8.3.5.7$
1^3	1^3	1^2	0001−	$+i/8.7\sqrt{3}$
1^3	1^3	2	0000+	$+1/8.7$
1^3	1^3	2	0001−	$-i/8.5.7\sqrt{3}$
21	1^3	1^2	0000+	$-1/3.5.7$
21	1^3	1^2	0001−	0
21	1^3	2	0000+	$-1/8.7\sqrt{5}$
21	1^3	2	0001−	$+i/8.7\sqrt{3.5}$
21	21	1^2	0000+	$-11/4.9.5.7\sqrt{2.3}$
21	21	1^2	0001−	$+i/4.9.7\sqrt{2}$
21	21	1^2	0010−	$-37i/8.4.9.7\sqrt{3}$
21	21	1^2	0011+	$-1/8.4.9$
21	21	2	0000+	$+1/3.5.7\sqrt{2.3}$
21	21	2	0001−	0
21	21	2	0010−	$+13.19i/8.8.2.3.5.7\sqrt{3}$
21	21	2	0011+	$+3/8.8.2.7$
2^41	1^3	1^2	0000+	$-1/4.3.7\sqrt{5}$
2^41	1^3	1^2	0001−	$-i/4.7\sqrt{3.5}$
2^41	2^41	1^2	0000+	$+1/4.9.7\sqrt{2.3}$
2^41	2^41	1^2	0001−	$+i/4.3.5\sqrt{2}$
2^41	2^41	1^2	0010−	$+5i/8.9.7\sqrt{3}$
2^41	2^41	1^2	0011+	$+1/8.3.5.7$
21^3	1^5	1^4	0000+	$+1/3.5.7$
21^3	1^5	1^4	0001−	0
21^3	21^3	1^4	0000+	$-29/8.8.9.3.5.7$
21^3	21^3	1^4	0001−	$-5i/8.8.7\sqrt{3}$
21^3	21^3	1^4	0010−	$+5i/8.8.7\sqrt{3}$
21^3	21^3	1^4	0011+	$+5/8.8.3.7$
21^3	21^3	2^5	0000+	$+19/8.8.9.7$
21^3	21^3	2^5	0001−	$+5i/8.8.7\sqrt{3}$
21^3	21^3	2^5	0010−	$-5i/8.8.7\sqrt{3}$
21^3	21^3	2^5	0011+	$-5/8.8.3.7$
2^21^3	1	21^4	0000+	$-19/8.4.3.5.7$
2^21^3	1	21^4	0001−	$-i\sqrt{3}/8.4.7$
2^21^3	1	21^4	0100−	$+i\sqrt{3}/8.4.7$
2^21^3	1	21^4	0101+	$+1/8.4.7$
2^21^3	2^21^3	0	0000+	$+1/4.3.7$
2^21^3	2^21^3	0	0001−	0
2^21^3	2^21^3	0	0010−	0
2^21^3	2^21^3	0	0011+	$-1/4.3.7$
2^21^3	2^21^3	21^4	0000+	$-1033/8.8.4.9.3.5.7$
2^21^3	2^21^3	21^4	0001−	$-i/8.8.4.3\sqrt{3}$
2^21^3	**21^3**	**21^4**		
2^21^3	2^21^3	21^4	0010−	$-i/8.8.4.3\sqrt{3}$
2^21^3	2^21^3	21^4	0011+	$+5.11/8.8.4.3.7$
2^21^3	2^21^3	21^4	0100−	$+i/8.8.4.3\sqrt{3}$
2^21^3	2^21^3	21^4	0101+	$+3/8.8.4.7$
2^21^3	2^21^3	21^4	0110+	$+3/8.8.4.7$
2^21^3	2^21^3	21^4	0111−	$+3i\sqrt{3}/8.8.4.7$
2^21^3	2^21^3	21^4	1000−	$+i/8.8.4.3\sqrt{3}$
2^21^3	2^21^3	21^4	1001+	$+3/8.8.4.7$
2^21^3	2^21^3	21^4	1010+	$+3/8.8.4.7$
2^21^3	2^21^3	21^4	1011−	$+3i\sqrt{3}/8.8.4.7$
2^21^3	2^21^3	21^4	1100+	$+5.11/8.8.4.3.7$
2^21^3	2^21^3	21^4	1101−	$-3i\sqrt{3}/8.8.4.7$
2^21^3	2^21^3	21^4	1110−	$-3i\sqrt{3}/8.8.4.7$
2^21^3	2^21^3	21^4	1111+	$-41/8.8.4.3.5.7$
3	**2^5**	**1^5**		
0	1	2^5	0000+	$-1/3\sqrt{2.7}$
1	2	1^5	0000+	$-1/3.7$
21^4	1	2^5	0000+	$+1/2.3.7\sqrt{2}$
3	**2^41**	**21^4**		
0	21^4	2^41	0000+	$+1/5.7\sqrt{2}$
1	1	2^5	0000+	$+1/7\sqrt{3.5}$
1^2	2	1^5	0000+	$-1/7\sqrt{2.3.5}$
2	2	1^5	0000+	$+1/2.7\sqrt{2.3.5}$
21^4	21^4	2^41	0000+	$+1/2.3.5.7\sqrt{3}$
21^4	21^4	2^41	0010−	$-i/2.3.5.7\sqrt{2.3}$
21^4	21^4	2^41	0100−	$-i/2.3.7\sqrt{2.3}$
21^4	21^4	2^41	0110+	$+17/8.3.5.7\sqrt{3}$
1^3	21	21^4	0000+	$+1/2.5.7$
21	21	21^4	0000+	$-1/9.3.5$
21	21	21^4	0010−	$+4i\sqrt{2}/9.3.5.7$
21	21	21^4	0100−	$-4i\sqrt{2}/9.3.5.7$
21	21	21^4	0110+	$+31/8.2.9.3.5.7$
2^21^3	1	2^5	0000+	$+1/4.7\sqrt{3.5}$
3	21	21^4	0000+	$-1/8.5.7$
3^5	**3**	**0**		
0	0	3	0000+	$-1/2\sqrt{2.7}$
1^5	1^5	2	0000+	$-1/4\sqrt{3.7}$
2^5	2^5	1	0000+	$-1/2.7\sqrt{2.3}$
21^4	21^4	21	0000+	$+1/2.7\sqrt{2.5}$
21^4	21^4	3	0000+	$+1/2.7\sqrt{2.5}$
2^41	2^41	21^4	0000+	$+1/4.7\sqrt{5}$
3^5	3^5	0	0000+	$-1/8.7$
3^5	**3**	**21^4**		
1^5	1^5	2	0000+	$+1/4.7$
2^5	2^5	1	0000+	$-1/4.7\sqrt{2}$
21^4	0	3	0000+	$+1/2.7\sqrt{2.5}$
21^4	21^4	21	0000+	$+\sqrt{3}/4.5.7$
21^4	21^4	21	0010−	$-i\sqrt{3}/4.5.7\sqrt{2}$
21^4	21^4	3	0000+	$+1/2.5.7\sqrt{3}$
21^4	21^4	3	0010−	$-i/5.7\sqrt{2.3}$
2^41	2^41	21^4	0000+	$+1/4.3.7$
2^41	2^41	21^4	0010−	$-i/2.3.5.7\sqrt{2}$
3^5	2^41	21^4	0000+	$+1/4.5.7$
3^5	3^5	0	0000+	$+1/8.7$
3^5	3^5	21^4	0000+	$-11/8.3.5.7$

Table 6—*continued*

$\mathbf{31^4}$	$\mathbf{2^5}$	$\mathbf{1}$		
0	1^5	2^5	0000+	$-1/3\sqrt{2.7}$
21^4	1^5	2^5	0000+	$+1/2.3.5.7\sqrt{2}$
21	2	1	0000+	$-1/3.5.7$
3	2	1	0000+	$-1/8.3.7$
$\mathbf{31^4}$	$\mathbf{21^4}$	$\mathbf{1^5}$		
0	1	21^4	0000+	$-1/\sqrt{2.3.5.7}$
1	2	1	0000+	$+1/7\sqrt{3.5}$
1^5	21^4	1^5	0000+	$-1/5.7$
1^4	1^5	2^5	0000+	$+1/3.5\sqrt{2.7}$
2^5	1^5	2^5	0000+	$-1/2.7\sqrt{2.3.5}$
21^4	1	21^4	0000+	$+1/2.5.7\sqrt{3}$
21^4	1	21^4	0100−	$+i/2.5.7\sqrt{2.3}$
21^3	21^4	1^5	0000+	$-1/2.3.5.7$
31^4	2	1	0000+	$-1/8.3.5.7$
32^4	21^4	1^5	0000+	$-1/8.5.7$
$\mathbf{31^4}$	$\mathbf{2^41}$	$\mathbf{1^2}$		
0	1^4	2^41	0000+	$-1/5\sqrt{2.3.7}$
1	1^5	2^5	0000+	$+\sqrt{2}/3.5\sqrt{7}$
1^2	21^4	1^5	0000+	$-1/5.7\sqrt{2}$
2	21^4	1^5	0000+	$+1/3.5.7$
21^4	1^4	2^41	0000+	$+1/2.9.5.7$
21^4	1^4	2^41	0100−	$-47\,i/8.9.5.7\sqrt{2}$
21^4	2^5	2^41	0000+	0
21^4	2^5	2^41	0100−	$+3\,i/4.5.7\sqrt{2}$
1^3	1	21^4	0000+	$-1/2.5\sqrt{2.3.7}$
21	1	21^4	0000+	$+\sqrt{2}/9.5.7$
21	1	21^4	0100−	$-i/4.9.5$
21^3	21	1^2	0000+	$+1/4.5.7$
2^21^3	1^5	2^5	0000+	$-1/4.5.7\sqrt{2.3}$
$\mathbf{31^4}$	$\mathbf{2^41}$	$\mathbf{2}$		
0	2^5	2^41	0000+	$+1/7\sqrt{2.3.5}$
1	1^5	2^5	0000+	$+1/7\sqrt{3.5}$
1^2	21^4	1^5	0000+	$+1/5.7\sqrt{2}$
2	21^4	1^5	0000+	$-1/2.5.7\sqrt{2}$
21^4	1^4	2^41	0000+	0
21^4	1^4	2^41	0100−	$+3\,i/4.5.7\sqrt{2}$
21^4	2^5	2^41	0000+	$+1/2.9.5.7\sqrt{2}$
21^4	2^5	2^41	0100−	$-37\,i/8.9.5.7$
21	1	21^4	0000+	$-1/3.5.7$
21	1	21^4	0100−	$-i/2.3.5.7\sqrt{2}$
3	1	21^4	0000+	$+1/8.7\sqrt{3.5}$
31^4	1^5	2^5	0000+	$-1/8.3.5.7$
$\mathbf{31^4}$	$\mathbf{21^3}$	$\mathbf{21^4}$		
0	21^4	21^3	0000+	$+1/2.7\sqrt{3.5}$
1	1	21^4	0000+	$+1/5.7$
1	2^21^3	21^4	0000+	$+1/8.3.5.7$
1^5	21^3	1^4	0000+	$-1/4.7\sqrt{3.5}$
1^2	1^2	2^21^3	0000+	$+1/4.7\sqrt{2.5}$
1^2	2	1	0000+	$+\sqrt{2}/5.7\sqrt{3}$
1^2	2	2^21^3	0000+	$+1/4.5\sqrt{3.7}$
1^4	1^4	2^41	0000+	$+1/4.5.7\sqrt{2}$
1^4	2^5	2^41	0000+	$+1/2.3.5.7\sqrt{2}$
2	1^2	2^21^3	0000+	$+1/4.3.7$
2	2	2^21^3	0000+	$+1/8\sqrt{3.5.7}$
2^5	1^4	2^41	0000+	$-1/8.5.7$
$\mathbf{31^4}$	$\mathbf{21^3}$	$\mathbf{21^4}$		
2^5	2^5	2^41	0000+	$-1/8.5.7\sqrt{2}$
21^4	21^4	1^5	0000+	$+1/2.5.7\sqrt{3}$
21^4	21^4	1^5	0010−	$+i\sqrt{3}/4.5.7\sqrt{2}$
21^4	21^4	21^3	0000+	$+31/8.4.3.5.7\sqrt{2}$
21^4	21^4	21^3	0010−	$+41\,i/8.8.2.5.7$
21^4	21^4	21^3	0100−	$-i/8.4.7\sqrt{2.3}$
21^4	21^4	21^3	0110+	$+5/8.8.2.7\sqrt{3}$
1^3	21	1^2	0000+	$+\sqrt{3}/4.5.7$
1^3	21	2	0000+	$-1/2.7\sqrt{2.3.5}$
21	21	1^2	0000+	$+1/3.5.7\sqrt{2}$
21	21	1^2	0010−	$-17\,i/8.4.3.5.7$
21	21	2	0000+	$+2/9.5.7$
21	21	2	0010−	$-11\,i/8.4.9.5\sqrt{2}$
21^3	1^5	2^5	0000+	$-1/2.3.7\sqrt{2.5}$
21^3	21^3	1^4	0000+	$+17/8.8.3.7\sqrt{3.5}$
21^3	21^3	1^4	0010−	$-i/8.8\sqrt{5}$
21^3	21^3	2^5	0000+	$-1/8.8.3\sqrt{5.7}$
21^3	21^3	2^5	0010−	$+i\sqrt{5}/8.8\sqrt{3.7}$
2^21^3	1	21^4	0000+	$+1/8.2.5.7\sqrt{2.3}$
2^21^3	1	21^4	0100−	$-i/8.2.7\sqrt{2}$
2^21^3	2^21^3	21^4	0000+	$+17.23/8.8.4.9.5.7$
2^21^3	2^21^3	21^4	0010−	$-i\sqrt{3}/8.8.4.7$
2^21^3	2^21^3	21^4	0100−	$+i\sqrt{3}/8.8.4.7$
2^21^3	2^21^3	21^4	0110+	$+1/8.8.4.7$
31^4	1	21^4	0000+	$-1/8.2.3.5.7$
31^4	2^21^3	21^4	0000+	$-17/8.4.3.5.7$
32^4	1^5	2^5	0000+	$+1/8.5\sqrt{3.7}$
32^4	21^3	1^4	0000+	$+1/8.8.3.7$
32^4	21^3	2^5	0000+	$+1/8.8.3.5$
$\mathbf{31^4}$	$\mathbf{2^21^3}$	$\mathbf{1^4}$		
0	1^2	2^21^3	0000+	$-1/2.3\sqrt{5.7}$
1	21	1^2	0000+	$+1/2.7\sqrt{2.3.5}$
1^5	1	21^4	0000+	$-1/2.3\sqrt{5.7}$
1^5	2^21^3	21^4	0000+	$-1/4.7\sqrt{3.5}$
1^2	1^4	2^41	0000+	$+1/2.5\sqrt{2.3.7}$
1^4	21^4	1^5	0000+	$-1/2.7\sqrt{3.5}$
1^4	21^4	21^3	0000+	$+1/4.7\sqrt{2.5}$
2^5	21^4	21^3	0000+	$+1/4.3.7$
21^4	1^2	2^21^3	0000+	$+\sqrt{3}/8.2.7\sqrt{2}$
21^4	1^2	2^21^3	0100−	$-i/8.2.3.5.7\sqrt{2}$
21^4	2	1	0000+	$-1/3.7\sqrt{2.5}$
21^4	2	2^21^3	0000+	$+1/8.2\sqrt{5.7}$
21^4	2	2^21^3	0100−	$+i/8.2\sqrt{3.5.7}$
1^3	1^5	2^5	0000+	$+1/3.5\sqrt{2.7}$
1^3	21^3	1^4	0000+	$+1/4.3.7$
1^3	21^3	2^5	0000+	$+1/4.5\sqrt{3.7}$
21	21^3	1^4	0000+	$+1/2.3.5.7$
2^41	1^5	2^5	0000+	$-1/4.3.7\sqrt{5}$
2^41	21^3	1^4	0000+	$-1/8.3.7$
2^41	21^3	2^5	0000+	$+1/8.2\sqrt{3.5.7}$
21^3	1	21^4	0000+	$-1/8.7\sqrt{2.5}$
21^3	1	21^4	0100−	$+i/8.7\sqrt{2.3.5}$
21^3	2^21^3	21^4	0000+	$+17/8.8.3.7\sqrt{3.5}$
21^3	2^21^3	21^4	0100−	$+i/8.8\sqrt{5}$
2^21^3	21	1^2	0000+	$-1/8.7\sqrt{5}$

Table 6—*continued*

31^4	**2^21^3**	**1^4**		
2^21^3	21	2	0000+	$-1/4.3.7\sqrt{2}$
31^4	21	1^2	0000+	$-11/8.4.3.5.7$
31^4	21	2	0000+	$-1/8.2\sqrt{2.3.5.7}$
31^4	**2^21^3**	**2^5**		
0	2	2^21^3	0000+	$+1/2.3.7$
1	21	1^2	0000+	$-1/2.3.5\sqrt{2.7}$
1^5	1	21^4	0000+	$+1/2.3\sqrt{5.7}$
1^4	21^4	1^5	0000+	$-1/3.5\sqrt{2.7}$
1^4	21^4	21^3	0000+	$+1/4.5\sqrt{3.7}$
2^5	21^4	21^3	0000+	$+1/8\sqrt{3.5.7}$
21^4	1^2	2^21^3	0000+	$+1/8.2\sqrt{5.7}$
21^4	1^2	2^21^3	0100−	$+i/8.2\sqrt{3.5.7}$
21^4	2	1	0000+	$-1/2.5\sqrt{2.3.7}$
21^4	2	2^21^3	0000+	$+\sqrt{3}/8.4.5$
21^4	2	2^21^3	0100−	$+i/8.4.3.7$
1^3	21^3	1^4	0000+	$+1/4.5\sqrt{3.7}$
1^3	21^3	2^5	0000+	$+1/4.3.7$
2^41	1^5	2^5	0000+	$-1/4.5\sqrt{3.7}$
2^41	21^3	1^4	0000+	$+1/8.2\sqrt{3.5.7}$
2^41	21^3	2^5	0000+	$-17/8.4.3.5.7$
21^3	2^21^3	21^4	0000+	$-1/8.8.3\sqrt{5.7}$
21^3	2^21^3	21^4	0100−	$-i\sqrt{5}/8.8\sqrt{3.7}$
2^21^3	21	1^2	0000+	$+\sqrt{3}/8.2.5\sqrt{7}$
2^21^3	21	2	0000+	$+1/8\sqrt{2.3.5.7}$
31^4	21	1^2	0000+	$-1/8.2\sqrt{2.3.5.7}$
31^4	21	2	0000+	$-1/8.2.3.5$
32^4	1	21^4	0000+	$+1/8.5\sqrt{3.7}$
31^4	**3^5**	**1^2**		
0	1^4	3^5	0000+	$-1/2\sqrt{2.3.5.7}$
1	1^5	2^5	0000+	$+1/4.3\sqrt{5}$
1	21^3	2^5	0000+	$+1/2.3.7\sqrt{5}$
2	21^4	1^5	0000+	$-1/2.3\sqrt{2.5.7}$
21^4	1^4	2^41	0000+	$+1/2.5\sqrt{2.3.7}$
21^4	1^4	3^5	0000+	$+1/2.5.7$
21^4	2^5	2^41	0000+	$-1/2.5\sqrt{2.3.7}$
21^4	2^5	3^5	0000+	$-1/2.5\sqrt{2.3.7}$
21	1	21^4	0000+	$+1/4.5\sqrt{3.7}$
21	2^21^3	21^4	0000+	$-1/4.5\sqrt{3.7}$
31^4	21^3	1^4	0000+	$+1/8.3.5$
31^4	21^3	2^5	0000+	$+1/8.7\sqrt{5}$
31^4	**3^5**	**2**		
0	2^5	3^5	0000+	$+1/2.7\sqrt{2.3}$
1	1^5	2^5	0000+	$+1/4.7$
2	21^4	1^5	0000+	$+\sqrt{3}/4.7\sqrt{2.5}$
21^4	1^4	2^41	0000+	$-1/2.5.7$
21^4	1^4	3^5	0000+	$-1/2.5\sqrt{2.3.7}$
21^4	2^5	2^41	0000+	$+3/4.5.7\sqrt{2}$
21^4	2^5	3^5	0000+	$+\sqrt{2}/3.5.7$
21	1	21^4	0000+	$+1/4.5.7$
21	2^21^3	21^4	0000+	$-3/8.5.7$
3	1	21^4	0000+	$-1/4.7\sqrt{3.5}$
31^4	1^5	2^5	0000+	$+1/4.3.5.7$
31^4	21^3	1^4	0000+	$+1/8.7\sqrt{5}$
31^4	21^3	2^5	0000+	$+1/8.3.5$
31^4	**31^4**	**1^4**		
1^5	1	21^4	0000+	$-1/4.5\sqrt{3}$
1^4	0	32^4	0000+	$-1/2.3.5\sqrt{2}$
21^4	1^2	2^21^3	0000+	$+1/8\sqrt{3.5.7}$
21^4	1^2	31^4	0000+	$-1/4.5\sqrt{2.11}$
21^4	1^2	31^4	0100−	$-i/4.3.5\sqrt{11}$
21^4	2	1	0000+	$+1/2.5\sqrt{2.3.7}$
21^4	2	2^21^3	0000+	$+1/4.3.5\sqrt{2}$
21^4	2	31^4	0000+	$-\sqrt{11}/4.3.5\sqrt{3.7}$
21^4	2	31^4	0100−	0
2^41	1^5	2^5	0000+	$+1/4.5\sqrt{3.7}$
21^3	1	21^4	0000+	$+1/4.3.5\sqrt{7}$
21^3	2^21^3	21^4	0000+	$+1/8.2.3.5\sqrt{2.3.7}$
2^21^3	21	1^2	0000+	$+1/8\sqrt{2.3.5.7}$
2^21^3	21	2	0000+	$+1/8.3.5$
31^4	21	1^2	0000+	$+1/8.2.3.5$
31^4	21	2	0000+	$+1/8.5\sqrt{2.3.7}$
31^4	3	1^2	0000+	$-1/8.3.5$
31^4	3	2	0000+	$-1/8.3.5$
32^4	2^21^3	21^4	0000+	$+1/8.4\sqrt{2.3.5.7.11}$
32^4	2^21^3	21^4	0100−	$+i\sqrt{7}/8.4\sqrt{3.5.11}$
32^4	31^4	0	0000+	$-1/8.3.5$
31^4	**31^4**	**2^5**		
1^5	1	21^4	0000+	$+\sqrt{3}/4.7\sqrt{5}$
2^5	0	32^4	0000+	$+1/2.3\sqrt{2.5.7}$
21^4	1^2	2^21^3	0000+	$+1/4.7\sqrt{2.5}$
21^4	1^2	31^4	0000+	$-\sqrt{11}/4.3.5\sqrt{3.7}$
21^4	1^2	31^4	0100−	0
21^4	2	1	0000+	$+3/4.5.7\sqrt{2}$
21^4	2	2^21^3	0000+	$+\sqrt{3}/8.5\sqrt{2.7}$
21^4	2	31^4	0000+	$-17/8.9.7\sqrt{11}$
21^4	2	31^4	0100−	$-i/4.3.5\sqrt{2.11}$
2^41	1^5	2^5	0000+	$-1/4.5.7$
21^3	2^21^3	21^4	0000+	$+1/8.2.3.7\sqrt{2.5}$
2^21^3	21	1^2	0000+	$-3/8.2.7\sqrt{2.5}$
2^21^3	21	2	0000+	$-\sqrt{3}/8.2.5\sqrt{7}$
3^5	1^5	2^5	0000+	$+1/4.7\sqrt{3.5}$
31^4	21	1^2	0000+	$+1/8.5\sqrt{2.3.7}$
31^4	21	2	0000+	$+11/8.9.5.7$
31^4	3	1^2	0000+	$-1/8.3.5$
31^4	3	2	0000+	$-59/8.9.3.5.7$
32^4	1	21^4	0000+	$+\sqrt{11}/4.3.5.7\sqrt{2}$
32^4	1	21^4	0100−	0
32^4	2^21^3	21^4	0000+	$-19/8.4.3.5\sqrt{2.3.7.11}$
32^4	2^21^3	21^4	0100−	$-i\sqrt{3}/8.4\sqrt{7.11}$
32^4	31^4	0	0000+	$+1/8.3.5$
32^4	**31^4**	**0**		
0	0	31^4	0000+	$-1/2\sqrt{2.3.5}$
1	1	2	0000+	$-1/4.3\sqrt{5}$
1^5	1^5	21^4	0000+	$-1/4.3\sqrt{5}$
1^2	1^2	21	0000+	$-1/2.3.5\sqrt{2}$
1^2	1^2	3	0000+	$-1/2.3.5\sqrt{2}$
1^4	1^4	21^3	0000+	$-1/2.3.5\sqrt{2}$
1^4	1^4	32^4	0000+	$-1/2.3.5\sqrt{2}$
2	2	21	0000+	$+1/2.3\sqrt{2.5.7}$
2	2	3	0000+	$+1/2.3\sqrt{2.5.7}$

Table 6—*continued*

32^4	31^4	0		
2^5	2^5	1^5	0000+	$-1/2.3\sqrt{2.5.7}$
2^5	2^5	21^3	0000+	$+1/2.3\sqrt{2.5.7}$
2^5	2^5	32^4	0000+	$+1/2.3\sqrt{2.5.7}$
21^4	21^4	1	0000+	$-1/2.5\sqrt{2.3.7}$
21^4	21^4	2^21^3	0000+	$+1/2.5\sqrt{2.3.7}$
21^4	21^4	31^4	0000+	$+1/2.5\sqrt{2.3.7}$
21^4	21^4	31^4	0100−	0
21^4	21^4	31^4	1000−	0
21^4	21^4	31^4	1100+	$-1/2.5\sqrt{2.3.7}$
2^41	2^41	1^4	0000+	$-1/4.5\sqrt{3.7}$
2^41	2^41	2^5	0000+	$+1/4.5\sqrt{3.7}$
21^3	21^3	21^4	0000+	$+1/4.3\sqrt{2.5.7}$
2^21^3	2^21^3	1^2	0000+	$-1/4.3\sqrt{2.5.7}$
2^21^3	2^21^3	2	0000+	$+1/4.3\sqrt{2.5.7}$
3^5	3^5	1^4	0000+	$-1/8\sqrt{3.5.7}$
3^5	3^5	2^5	0000+	$+1/8\sqrt{3.5.7}$
31^4	31^4	1^2	0000+	$-1/8.3.5$
31^4	31^4	2	0000+	$+1/8.3.5$
32^4	32^4	0	0000+	$-1/8.3.5$
32^4	31^4	21^4		
1	1	2	0000+	$-3/2.5.7\sqrt{2.11}$
1	1	2	0001−	$+7i/4.3.5\sqrt{11}$
1^5	1^5	21^4	0000+	$-9/2.5.7\sqrt{2.11}$
1^5	1^5	21^4	0001−	$-i/4.3.5\sqrt{11}$
1^2	1^2	21	0000+	$-9/8.5.7\sqrt{2.11}$
1^2	1^2	21	0001−	$+i/8.3\sqrt{11}$
1^2	1^2	3	0000+	0
1^2	1^2	3	0001−	$+i\sqrt{11}/2.3.5.7$
1^4	1^4	21^3	0000+	$-3/8.7\sqrt{2.11}$
1^4	1^4	21^3	0001−	$+i/8.3.5\sqrt{11}$
1^4	1^4	32^4	0000+	$-1/4.5\sqrt{2.11}$
1^4	1^4	32^4	0001−	$-i/4.3.5\sqrt{11}$
2	1^2	21	0000+	$+1/4.5.7\sqrt{2.11}$
2	1^2	21	0001−	$-i/4.5\sqrt{11}$
2	1^2	3	0000+	$+1/2.5\sqrt{3.7.11}$
2	1^2	3	0001−	$-i\sqrt{3}/2.5\sqrt{2.7.11}$
2	2	21	0000+	$-1/8.7\sqrt{11}$
2	2	21	0001−	$+i/3.5\sqrt{2.11}$
2	2	3	0000+	$-1/2.3.5.7\sqrt{11}$
2	2	3	0001−	$+i/3.5\sqrt{2.11}$
2^5	1^4	21^3	0000+	$-\sqrt{3}/8\sqrt{5.7.11}$
2^5	1^4	21^3	0001−	$-i/4\sqrt{2.3.5.7.11}$
2^5	1^4	32^4	0000+	$-\sqrt{11}/4.3.5\sqrt{3.7}$
2^5	1^4	32^4	0001−	0
2^5	2^5	1^5	0000+	$+9/4.5.7\sqrt{11}$
2^5	2^5	1^5	0001−	$+i/4.3.5\sqrt{2.11}$
2^5	2^5	21^3	0000+	$-3/8.2.5\sqrt{11}$
2^5	2^5	21^3	0001−	$+i/8.3.5.7\sqrt{2.11}$
2^5	2^5	32^4	0000+	$-17/8.9.7\sqrt{11}$
2^5	2^5	32^4	0001−	$-i/4.3.5\sqrt{2.11}$
21^4	0	31^4	0000+	$+1/2.5\sqrt{2.3.7}$
21^4	0	31^4	0001−	0
21^4	0	31^4	0100−	0
21^4	0	31^4	0101+	$-1/2.5\sqrt{2.3.7}$
21^4	21^4	1	0000+	$+\sqrt{3}/5.7\sqrt{2.11}$
32^4	31^4	21^4		
21^4	21^4	1	0001−	$-i/4.5\sqrt{3.11}$
21^4	21^4	1	0010−	$-3i\sqrt{3}/8.5.7\sqrt{11}$
21^4	21^4	1	0011+	$+1/2.5\sqrt{2.3.11}$
21^4	21^4	2^21^3	0000+	$-43/8.2.5.7\sqrt{2.3.11}$
21^4	21^4	2^21^3	0001−	$-i/8.2.5\sqrt{3.11}$
21^4	21^4	2^21^3	0010−	$+29i\sqrt{3}/8.8.5.7\sqrt{11}$
21^4	21^4	2^21^3	0011+	$-1/8.4.5\sqrt{2.3.11}$
21^4	21^4	31^4	0000+	$-3079/8.2.9.5.7.11\sqrt{2.3.11}$
21^4	21^4	31^4	0001−	$-i/8.2.11\sqrt{3.11}$
21^4	21^4	31^4	0010−	$+3433i/8.8.3.5.7.11\sqrt{3.11}$
21^4	21^4	31^4	0011+	$-29/8.4.5.11\sqrt{2.3.11}$
21^4	21^4	31^4	0100−	$-i/8.2.11\sqrt{3.11}$
21^4	21^4	31^4	0101+	$+3\sqrt{3}/8.5.11\sqrt{2.11}$
21^4	21^4	31^4	0110+	$-29/8.4.5.11\sqrt{2.3.11}$
21^4	21^4	31^4	0111−	$+9i\sqrt{3}/8.4.5.11\sqrt{11}$
21^4	21^4	31^4	1000−	$-i/8.2.11\sqrt{3.11}$
21^4	21^4	31^4	1001+	$+3\sqrt{3}/8.5.11\sqrt{2.11}$
21^4	21^4	31^4	1010+	$-29/8.4.5.11\sqrt{2.3.11}$
21^4	21^4	31^4	1011−	$+9i\sqrt{3}/8.4.5.11\sqrt{11}$
21^4	21^4	31^4	1100+	$+3\sqrt{3}/8.5.11\sqrt{2.11}$
21^4	21^4	31^4	1101−	$-59i\sqrt{3}/8.5.7.11\sqrt{11}$
21^4	21^4	31^4	1110−	$+9i\sqrt{3}/8.4.5.11\sqrt{11}$
21^4	21^4	31^4	1111+	$+443/8.2.5.7.11\sqrt{2.3.11}$
2^41	2^41	1^4	0000+	$+3/2.5.7\sqrt{2.11}$
2^41	2^41	1^4	0001−	$+i/4.9.5\sqrt{11}$
2^41	2^41	1^4	0010−	$-3i/8.8.5\sqrt{11}$
2^41	2^41	1^4	0011+	$-13/8.4.9.5\sqrt{2.11}$
2^41	2^41	2^5	0000+	$-3/2.5.7\sqrt{2.11}$
2^41	2^41	2^5	0001−	$-i/4.9.5\sqrt{11}$
2^41	2^41	2^5	0010−	$+9i/8.4.5.7\sqrt{11}$
2^41	2^41	2^5	0011+	$-7/8.2.9.5\sqrt{2.11}$
21^3	1^5	21^4	0000+	$+1/8.5.7\sqrt{2.11}$
21^3	1^5	21^4	0001−	$-i/8.5\sqrt{11}$
21^3	21^3	21^4	0000+	$+17\sqrt{3}/8.8.5.7\sqrt{11}$
21^3	21^3	21^4	0001−	$+i/8.5\sqrt{2.3.11}$
21^3	21^3	21^4	0010−	$+5i/8.8.7\sqrt{11}$
21^3	21^3	21^4	0011+	$+1/8.3\sqrt{2.11}$
2^21^3	1	2	0000+	$-\sqrt{3}/4.5\sqrt{2.7.11}$
2^21^3	1	2	0001−	$-i/4.5\sqrt{3.7.11}$
2^21^3	2^21^3	1^2	0000+	$-\sqrt{3}/8.8.2.7\sqrt{11}$
2^21^3	2^21^3	1^2	0001−	$+i\sqrt{3}/8.8\sqrt{2.11}$
2^21^3	2^21^3	1^2	0010−	$+9.13i/8.8.2.5.7\sqrt{11}$
2^21^3	2^21^3	1^2	0011+	$-1/8.8.3.5\sqrt{2.11}$
2^21^3	2^21^3	2	0000+	$-\sqrt{3}/8.8.2.5\sqrt{11}$
2^21^3	2^21^3	2	0001−	$-3.13i\sqrt{3}/8.8.5.7\sqrt{2.11}$
2^21^3	2^21^3	2	0010−	$-3i/8.8.2\sqrt{11}$
2^21^3	2^21^3	2	0011+	$-1/8.8.3.7\sqrt{2.11}$
3^5	2^41	1^4	0000+	$+\sqrt{3}/8.5\sqrt{7.11}$
3^5	2^41	1^4	0001−	$+i/4.5\sqrt{2.3.7.11}$
3^5	2^41	2^5	0000+	$-\sqrt{11}/4.3.5.7\sqrt{2}$
3^5	2^41	2^5	0001−	0
3^5	3^5	1^4	0000+	$-1/4.5\sqrt{2.11}$
3^5	3^5	1^4	0001−	$-i/8.5.7\sqrt{11}$
3^5	3^5	2^5	0000+	$+59/4.9.5.7\sqrt{2.11}$
3^5	3^5	2^5	0001−	$+i/8.3.5\sqrt{11}$

Table 6—*continued*

32^4	31^4	21^4		
31^4	1	2	0000+	$+\sqrt{11}/4.3.5.7\sqrt{2}$
31^4	1	2	0001−	0
31^4	2^21^3	1^2	0000+	$+1/8.4\sqrt{2.3.5.7.11}$
31^4	2^21^3	1^2	0001−	$-i\sqrt{7}/8.4\sqrt{3.5.11}$
31^4	2^21^3	2	0000+	$-19/8.4.3.5\sqrt{2.3.7.11}$
31^4	2^21^3	2	0001−	$+i\sqrt{3}/8.4\sqrt{7.11}$
31^4	31^4	1^2	0000+	$+53/8.4.9.5.11$
31^4	31^4	1^2	0001−	$-7i/8.2.5.11\sqrt{2}$
31^4	31^4	1^2	0010−	$+7i/8.2.5.11\sqrt{2}$
31^4	31^4	1^2	0011+	$-7/8.2.3.5.11$
31^4	31^4	2	0000+	$-2371/8.4.9.9.5.7.11$
31^4	31^4	2	0001−	$+7i/8.2.5.11\sqrt{2}$
31^4	31^4	2	0010−	$-7i/8.2.5.11\sqrt{2}$
31^4	31^4	2	0011+	$+7/8.2.3.5.11$
32^4	1^5	21^4	0000+	$+53/8.2.3.5.7.11$
32^4	1^5	21^4	0001−	$-3i/8.5.11\sqrt{2}$
32^4	1^5	21^4	0100−	$+3i/8.5.11\sqrt{2}$
32^4	1^5	21^4	0101+	$-1/8.5.11$
32^4	21^3	21^4	0000+	$+1051/8.8.9.5.7.11$
32^4	21^3	21^4	0001−	$+3i/8.4.5.11\sqrt{2}$
32^4	21^3	21^4	0100−	$-3i/8.4.5.11\sqrt{2}$
32^4	31^4	21^4		
32^4	21^3	21^4	0101+	$+1/8.4.5.11$
32^4	32^4	0	0000+	$+1/8.3.5$
32^4	32^4	0	0001−	0
32^4	32^4	0	0010−	0
32^4	32^4	0	0011+	$-1/8.3.5$
32^4	32^4	21^4	0000+	$-6829/4.9.9.5.7.11.11$
32^4	32^4	21^4	0001−	$+17i/8.2.9.11.11\sqrt{2}$
32^4	32^4	21^4	0010−	$+17i/8.2.9.11.11\sqrt{2}$
32^4	32^4	21^4	0011+	$+2.7/3.5.11.11$
32^4	32^4	21^4	0100−	$-17i/8.2.9.11.11\sqrt{2}$
32^4	32^4	21^4	0101+	$+3/8.5.11.11$
32^4	32^4	21^4	0110+	$+3/8.5.11.11$
32^4	32^4	21^4	0111−	$-9i/8.5.11.11\sqrt{2}$
32^4	32^4	21^4	1000−	$-17i/8.2.9.11.11\sqrt{2}$
32^4	32^4	21^4	1001+	$+3/8.5.11.11$
32^4	32^4	21^4	1010+	$+3/8.5.11.11$
32^4	32^4	21^4	1011−	$-9i/8.5.11.11\sqrt{2}$
32^4	32^4	21^4	1100+	$+2.7/3.5.11.11$
32^4	32^4	21^4	1101−	$+9i/8.5.11.11\sqrt{2}$
32^4	32^4	21^4	1110−	$+9i/8.5.11.11\sqrt{2}$
32^4	32^4	21^4	1111+	$-19/8.3.5.7.11.11$

Table 7. Branching rules for $SU_3 \supset SO_3$.

0	→	+ 0		
1	→	+ 1		
1^2	→	+ 1		
2	→	+ 0	+ 2	
2^2	→	+ 0	+ 2	
21	→	+ 1	+ 2	
3	→	+ 1	+ 3	
3^2	→	+ 1	+ 3	
31	→	+ 1	+ 2	+ 3
32	→	+ 1	+ 2	+ 3

Table 8. $3jm$ factors for $SU_3 \supset SO_3$.

0	**0**	**0**	**0**	
0	0	0	+	$+1$
1	**1**	**1**	**0**	
1	1	1	+	$+1$
$\mathbf{1^2}$	**1**	**0**	**0**	
1	1	0	+	$+1$
2	$\mathbf{1^2}$	$\mathbf{1^2}$	**0**	
0	1	1	+	$+1/\sqrt{2.3}$
2	1	1	+	$+\sqrt{5}/\sqrt{2.3}$
2	**2**	**2**	**0**	
0	0	0	+	$+\sqrt{2}/3\sqrt{3}$
2	2	0	+	$-\sqrt{5}/3\sqrt{2.3}$
2	2	2	+	$+\sqrt{5.7}/3\sqrt{2.3}$
$\mathbf{2^2}$	**2**	**0**	**0**	
0	0	0	+	$+1/\sqrt{2.3}$
2	2	0	+	$+\sqrt{5}/\sqrt{2.3}$
21	$\mathbf{1^2}$	**1**	**0**	
1	1	1	−	$+i\sqrt{3}/2\sqrt{2}$
21	$\mathbf{1^2}$	**1**	**0**	
2	1	1	+	$+\sqrt{5}/2\sqrt{2}$
21	**2**	**1**	**0**	
1	0	1	−	$-i/\sqrt{2.3}$
1	2	1	−	$+i\sqrt{5}/2\sqrt{2.3}$
2	2	1	+	$-\sqrt{5}/2\sqrt{2}$
21	$\mathbf{2^2}$	**2**	**0**	
1	2	2	−	$+i\sqrt{3}/2\sqrt{2}$
2	0	2	+	$+1/\sqrt{2.3}$

Table 8—*continued*

21	**2^2**	**2**	**0**	
2	2	2	+	$+\sqrt{7}/2\sqrt{2.3}$
21	**21**	**0**	**0**	
1	1	0	+	$+\sqrt{3}/2\sqrt{2}$
2	2	0	+	$+\sqrt{5}/2\sqrt{2}$
21	**21**	**21**	**0**	
1	1	1	−	0
2	1	1	+	$-\sqrt{3}/4$
2	2	1	−	0
2	2	2	+	$+\sqrt{7}/4$
21	**21**	**21**	**1**	
1	1	1	+	$-1/4$
2	1	1	−	0
2	2	1	+	$-\sqrt{5}/4$
2	2	2	−	0
3	**2^2**	**1^2**	**0**	
1	0	1	+	$+1/\sqrt{2.3}$
1	2	1	+	$+\sqrt{2}/\sqrt{3.5}$
3	2	1	+	$+\sqrt{7}/\sqrt{2.5}$
3	**21**	**21**	**0**	
1	1	1	+	$-1/2\sqrt{2}$
1	2	1	−	$+i\sqrt{3}/2\sqrt{2.5}$
1	2	2	+	$+1/2\sqrt{2.5}$
3	2	1	−	$-i\sqrt{7}/2\sqrt{2.5}$
3	2	2	+	$+\sqrt{7}/2\sqrt{5}$
3	**3**	**3**	**0**	
1	1	1	+	$+2/5$
3	3	1	+	$-\sqrt{7}/5\sqrt{2}$
3	3	3	+	$+\sqrt{3.7}/5\sqrt{2}$
3^2	**3**	**0**	**0**	
1	1	0	+	$+\sqrt{3}/\sqrt{2.5}$
3	3	0	+	$+\sqrt{7}/\sqrt{2.5}$
3^2	**3**	**21**	**0**	
1	1	1	−	$+i/2\sqrt{2.5}$
1	1	2	+	$+3\sqrt{3}/2.5\sqrt{2}$
1	3	2	+	$+\sqrt{7}/5\sqrt{2}$
3	3	1	−	$+i\sqrt{7}/2\sqrt{5}$
3	3	2	+	$+\sqrt{3.7}/2.5$
31	**2^2**	**1**	**0**	
1	0	1	+	$+1/\sqrt{2.3}$
1	2	1	+	$-1/\sqrt{2.3.5}$
2	2	1	−	$+1/\sqrt{3}$
3	2	1	+	$+\sqrt{7}/\sqrt{3.5}$
31	**2^2**	**2^2**	**0**	
1	2	2	+	$-1/\sqrt{5}$
2	2	0	−	$-1/\sqrt{2.3}$
2	2	2	−	0
3	2	2	+	$-\sqrt{7}/\sqrt{3.5}$
31	**21**	**1^2**	**0**	
1	1	1	−	$+i/2\sqrt{2}$
1	2	1	+	$-\sqrt{3}/2\sqrt{2.5}$
2	1	1	+	$+i/2$
2	2	1	−	$+1/2\sqrt{3}$
3	2	1	+	$-\sqrt{7}/\sqrt{3.5}$
31	**21**	**2**	**0**	
1	1	0	−	$+i/3\sqrt{2}$
31	**21**	**2**	**0**	
1	1	2	−	$+7i/2.3\sqrt{2.5}$
1	2	2	+	$+1/2\sqrt{2.3.5}$
2	1	2	+	$-i/2.3$
2	2	0	+	$+1/3$
2	2	2	−	$-\sqrt{7}/2.3$
3	1	2	−	$-i\sqrt{7}/3\sqrt{5}$
3	2	2	+	$-\sqrt{2.7}/3\sqrt{5}$
31	**3**	**2**	**0**	
1	1	0	+	$+2/3\sqrt{5}$
1	1	2	+	$-2/3.5$
1	3	2	+	$-\sqrt{7}/5\sqrt{3}$
2	1	2	−	$+2\sqrt{2}/3\sqrt{5}$
2	3	2	−	$-\sqrt{7}/3\sqrt{5}$
3	1	2	+	$-\sqrt{7}/3.5\sqrt{2}$
3	3	0	+	$-\sqrt{7}/3\sqrt{2.5}$
3	3	2	+	$+2\sqrt{7}/5\sqrt{3}$
31	**3^2**	**1^2**	**0**	
1	1	1	+	$-1/\sqrt{5}$
2	1	1	−	$+1/\sqrt{2.5}$
2	3	1	−	$-\sqrt{7}/\sqrt{2.3.5}$
3	3	1	+	$-\sqrt{7}/\sqrt{3.5}$
31	**3^2**	**2**	**0**	
1	1	0	+	$+1/3\sqrt{2.5}$
1	1	2	+	$+4\sqrt{2}/3.5$
1	3	2	+	$-\sqrt{7}/5\sqrt{2.3}$
2	1	2	−	$+1/3\sqrt{5}$
2	3	2	−	$+\sqrt{2.7}/3\sqrt{5}$
3	1	2	+	$+2\sqrt{7}/3.5$
3	3	0	+	$+\sqrt{7}/3\sqrt{5}$
3	3	2	+	$+\sqrt{2.7}/5\sqrt{3}$
31	**31**	**1**	**0**	
1	1	1	+	$+1/\sqrt{3.5}$
2	1	1	−	$-\sqrt{2}/\sqrt{3.5}$
2	2	1	+	$+2/3\sqrt{3}$
3	2	1	−	$-\sqrt{7}/3\sqrt{3.5}$
3	3	1	+	$-2\sqrt{2.7}/3\sqrt{3.5}$
31	**31**	**2^2**	**0**	
1	1	0	+	$-1/3\sqrt{2.3.5}$
1	1	2	+	$+7/3.5\sqrt{2.3}$
2	1	2	−	$+1/\sqrt{3.5}$
2	2	0	+	$-\sqrt{2}/9$
2	2	2	+	$-\sqrt{2.7}/9$
3	1	2	+	$+\sqrt{7}/5\sqrt{3}$
3	2	2	−	$+2\sqrt{7}/9\sqrt{5}$
3	3	0	+	$+2\sqrt{2.7}/9\sqrt{5}$
3	3	2	+	$+4\sqrt{7}/3.5\sqrt{3}$
31	**31**	**31**	**0**	
1	1	1	+	$-\sqrt{2}/5$
2	1	1	−	0
2	2	1	+	$+2\sqrt{2}/9\sqrt{5}$
2	2	2	−	0
3	2	1	−	$-\sqrt{7}/9\sqrt{2}$
3	2	2	+	$-\sqrt{2.7}/3\sqrt{3.5}$
3	3	1	+	$+2\sqrt{7}/9.5$
3	3	2	−	0
31	**31**	**31**	**0**	
3	3	3	+	$-2\sqrt{2.7}/3.5$
31	**31**	**31**	**1**	
1	1	1	−	0
2	1	1	+	$-2/5\sqrt{3}$
2	2	1	−	0
2	2	2	+	0
3	2	1	+	$-\sqrt{7}/5\sqrt{2.3}$
3	2	2	−	0
3	3	1	−	0
3	3	2	+	$+\sqrt{2.7}/5\sqrt{3}$
3	3	3	−	0
32	**31**	**0**	**0**	
1	1	0	+	$+1/\sqrt{5}$
2	2	0	+	$+1/\sqrt{3}$
3	3	0	+	$+\sqrt{7}/\sqrt{3.5}$
32	**31**	**21**	**0**	
1	1	1	−	$-i/2\sqrt{2.3.5}$
1	1	2	+	$+1/2.5\sqrt{2}$
1	2	1	+	$+i/2\sqrt{3.5}$
1	2	2	−	$+\sqrt{5}/2.3$
1	3	2	+	$+\sqrt{7}/3.5$
2	2	1	−	$+i\sqrt{2}/3\sqrt{3}$
2	2	2	+	0
2	3	1	+	$-i\sqrt{2.7}/3\sqrt{3.5}$
2	3	2	−	0
3	3	1	−	$+i\sqrt{7}/3\sqrt{3.5}$
3	3	2	+	$+\sqrt{7}/5$
32	**31**	**21**	**1**	
1	1	1	+	$-\sqrt{2}/3\sqrt{5}$
1	1	2	−	$-2i\sqrt{2}/5\sqrt{3}$
1	2	1	−	$+1/2.3\sqrt{5}$
1	2	2	+	$+i/2.3\sqrt{3.5}$
1	3	2	−	$+2i\sqrt{7}/3.5\sqrt{3}$
2	2	1	+	$-5/2.9\sqrt{2}$
2	2	2	−	$+i\sqrt{7}/2.3\sqrt{2}$
2	3	1	−	$-\sqrt{2.7}/9\sqrt{5}$
2	3	2	+	$-i\sqrt{7}/3\sqrt{5}$
3	3	1	+	$-7\sqrt{7}/2.9\sqrt{5}$
3	3	2	−	$-i\sqrt{7}/2.5\sqrt{3}$
32	**31**	**3**	**0**	
1	1	1	+	$-1/3.5$
1	2	1	−	$+\sqrt{2}/3.5$
1	2	3	−	$-\sqrt{2.7}/3.5\sqrt{3}$
1	3	3	+	$+4\sqrt{7}/3.5\sqrt{3}$
2	1	1	−	$-1/3\sqrt{2}$
2	1	3	−	$-\sqrt{7}/3\sqrt{2.3}$
2	2	1	+	$-\sqrt{5}/9$
2	2	3	+	0
2	3	1	−	$+\sqrt{7}/9$
2	3	3	−	0
3	1	3	+	$-\sqrt{7}/3.5\sqrt{3}$
3	2	1	−	$+4\sqrt{7}/9.5$
3	2	3	−	$-\sqrt{2.7}/5\sqrt{3}$
3	3	1	+	$+2\sqrt{2.7}/9.5$
3	3	3	+	$+\sqrt{2.7}/5\sqrt{3}$

Table 9. Branching rules for $SU_3 \supset U_1 \times SU_2$.

0	->	+ 0.0					
1	->	+ -2.0	+ 1.1				
1^2	->	+ 2.0	− -1.1				
2	->	+ -4.0	+ -1.1	+ 2.2			
2^2	->	+ 4.0	− 1.1	+ -2.2			
21	->	+ 0.0	+ 3.1	− -3.1	+ 0.2		
3	->	+ -6.0	+ -3.1	+ 0.2	+ 3.3		
3^2	->	+ 6.0	− 3.1	+ 0.2	− -3.3		
31	->	+ -2.0	+ 1.1	+ -5.1	+ -2.2	+ 4.2	+ 1.3
32	->	+ 2.0	− -1.1	− 5.1	+ 2.2	+ -4.2	− -1.3

Table 10. $3jm$ factors for $SU_3 \supset U_1 \times SU_2$.

0	**0**	**0**	**0**	
0.0	0.0	0.0	+	$+1$
1	**1**	**1**	**0**	
1.1	1.1	-2.0	+	$+1/\sqrt{3}$
1^2	**1**	**0**	**0**	
2.0	-2.0	0.0	+	$+1/\sqrt{3}$
-1.1	1.1	0.0	−*	$-\sqrt{2}/\sqrt{3}$
2	**1^2**	**1^2**	**0**	
-4.0	2.0	2.0	+	$+1/\sqrt{2.3}$
-1.1	-1.1	2.0	−*	$+1/\sqrt{2.3}$
2.2	-1.1	-1.1	+	$+1/\sqrt{2}$
2	**2**	**2**	**0**	
2.2	-1.1	-1.1	+	$-1/\sqrt{2.3}$
2.2	2.2	-4.0	+	$+1/\sqrt{2.3}$
2^2	**2**	**0**	**0**	
4.0	-4.0	0.0	+	$+1/\sqrt{2.3}$
1.1	-1.1	0.0	−*	$-1/\sqrt{3}$
-2.2	2.2	0.0	+	$+1/\sqrt{2}$
21	**1^2**	**1**	**0**	
0.0	2.0	-2.0	+	$+1/2\sqrt{3}$
0.0	-1.1	1.1	−*	$+1/2\sqrt{2.3}$
3.1	-1.1	-2.0	−*	$+1/2$
0.2	-1.1	1.1	+*	$+i\sqrt{3}/2\sqrt{2}$
21	**2**	**1**	**0**	
0.0	-1.1	1.1	+	$-1/2\sqrt{2}$
3.1	-4.0	1.1	+	$-1/\sqrt{2.3}$
3.1	-1.1	-2.0	+	$-1/2\sqrt{3}$
-3.1	2.2	1.1	−*	$-1/2$
0.2	-1.1	1.1	−	$+i/2\sqrt{2}$
0.2	2.2	-2.0	−	$-i/2$
21	**2^2**	**2**	**0**	
0.0	4.0	-4.0	+	$+1/\sqrt{3.5}$
0.0	1.1	-1.1	−*	$-1/2\sqrt{2.3.5}$
0.0	-2.2	2.2	+	$-1/2\sqrt{5}$
3.1	1.1	-4.0	−*	$+1/\sqrt{2.5}$
21	**2^2**	**2**	**0**	
3.1	-2.2	-1.1	+	$+\sqrt{3}/2\sqrt{5}$
0.2	1.1	-1.1	+*	$+i\sqrt{3}/2\sqrt{2.5}$
0.2	-2.2	2.2	−	$-i\sqrt{3}/\sqrt{2.5}$
21	**21**	**0**	**0**	
0.0	0.0	0.0	+	$+1/2\sqrt{2}$
-3.1	3.1	0.0	−*	$-1/2$
0.2	0.2	0.0	+	$+\sqrt{3}/2\sqrt{2}$
21	**21**	**21**	**0**	
0.0	0.0	0.0	+	$+1/2\sqrt{2.5}$
-3.1	3.1	0.0	−*	$-1/4\sqrt{5}$
0.2	-3.1	3.1	+*	$+3i/4\sqrt{5}$
0.2	0.2	0.0	+	$-\sqrt{3}/2\sqrt{2.5}$
0.2	0.2	0.2	−	0
21	**21**	**21**	**1**	
0.0	0.0	0.0	−	0
-3.1	3.1	0.0	+*	$+i/4$
0.2	-3.1	3.1	−*	$-1/4$
0.2	0.2	0.0	−	0
0.2	0.2	0.2	+	$+1/2$
3	**2^2**	**1^2**	**0**	
-6.0	4.0	2.0	+	$+1/\sqrt{2.5}$
-3.1	4.0	-1.1	−*	$+1/\sqrt{3.5}$
-3.1	1.1	2.0	−*	$-\sqrt{2}/\sqrt{3.5}$
0.2	1.1	-1.1	+	$+1/\sqrt{5}$
0.2	-2.2	2.0	+	$+1/\sqrt{2.5}$
3.3	-2.2	-1.1	−*	$+\sqrt{2}/\sqrt{5}$
3	**21**	**21**	**0**	
-6.0	3.1	3.1	+	$+1/\sqrt{2.5}$
-3.1	3.1	0.0	+	$-1/2\sqrt{5}$
-3.1	0.2	3.1	−	$-i/2\sqrt{5}$
0.2	-3.1	3.1	−*	$-1/2\sqrt{5}$
0.2	0.2	0.0	−	$-i\sqrt{3}/2\sqrt{2.5}$
0.2	0.2	0.2	+	$-1/2\sqrt{5}$
3.3	0.2	-3.1	+*	$-i/\sqrt{5}$
3	**3**	**3**	**0**	
0.2	0.2	0.2	+	$-1/\sqrt{2.5}$
3.3	0.2	-3.1	+	$+1/\sqrt{2.5}$
3.3	3.3	-6.0	+	$+1/\sqrt{2.5}$
3^2	**3**	**0**	**0**	
6.0	-6.0	0.0	+	$+1/\sqrt{2.5}$
3.1	-3.1	0.0	−*	$-1/\sqrt{5}$
0.2	0.2	0.0	+	$+\sqrt{3}/\sqrt{2.5}$
-3.3	3.3	0.0	−*	$-\sqrt{2}/\sqrt{5}$
3^2	**3**	**21**	**0**	
6.0	-6.0	0.0	+	$+1/2\sqrt{5}$
6.0	-3.1	-3.1	−*	$-1/2\sqrt{5}$
3.1	-3.1	0.0	−*	$-1/2\sqrt{2.5}$
3.1	-3.1	0.2	+*	$+i/2\sqrt{2.5}$
3.1	0.2	-3.1	+	$-1/\sqrt{2.5}$
0.2	0.2	0.0	+	0
0.2	0.2	0.2	−	$-i/\sqrt{2.5}$
0.2	3.3	-3.1	−*	$-1/\sqrt{2.5}$
-3.3	3.3	0.0	−*	$+1/2\sqrt{5}$
-3.3	3.3	0.2	+*	$+i/2$
31	**2^2**	**1**	**0**	
-2.0	4.0	-2.0	+	$+1/\sqrt{2.3.5}$
-2.0	1.1	1.1	−*	$+1/\sqrt{2.3.5}$
1.1	1.1	-2.0	−*	$+1/\sqrt{2.5}$
1.1	-2.2	1.1	+	$-1/\sqrt{2.3.5}$
-5.1	4.0	1.1	−	$+\sqrt{2}/\sqrt{3.5}$
-2.2	1.1	1.1	+*	$+1/\sqrt{5}$
4.2	-2.2	-2.0	+	$+1/\sqrt{5}$
1.3	-2.2	1.1	−	$+2/\sqrt{3.5}$
31	**2^2**	**2^2**	**0**	
-2.0	1.1	1.1	+	$-1/\sqrt{3.5}$
1.1	-2.2	1.1	−*	$+1/\sqrt{3.5}$
-5.1	1.1	4.0	+*	$-1/\sqrt{3.5}$
-2.2	1.1	1.1	−	0
-2.2	-2.2	4.0	−	$+1/\sqrt{2.5}$

Table 10—*continued*

31	**2^2**	**2^2**	**0**	
4.2	-2.2	-2.2	+	$-1/\sqrt{5}$
1.3	-2.2	1.1	+*	$+\sqrt{2}/\sqrt{3.5}$
31	**21**	**1^2**	**0**	
-2.0	0.0	2.0	+	$-1/2\sqrt{5}$
-2.0	3.1	-1.1	−*	$+1/2\sqrt{3.5}$
1.1	0.0	-1.1	−*	$+\sqrt{3}/2\sqrt{2.5}$
1.1	-3.1	2.0	−*	$-1/2\sqrt{5}$
1.1	0.2	-1.1	+*	$+i/2\sqrt{2.3.5}$
-5.1	3.1	2.0	−	$+\sqrt{2}/\sqrt{3.5}$
-2.2	3.1	-1.1	+*	$-1/\sqrt{2.5}$
-2.2	0.2	2.0	+	$+i/\sqrt{2.5}$
4.2	-3.1	-1.1	+	$-1/\sqrt{5}$
1.3	0.2	-1.1	−*	$+2i/\sqrt{3.5}$
31	**21**	**2**	**0**	
-2.0	3.1	-1.1	+	$-1/2\sqrt{5}$
-2.0	0.2	2.2	−	$+i/2\sqrt{3.5}$
1.1	0.0	-1.1	+	$-1/2\sqrt{2.5}$
1.1	3.1	-4.0	+	$-1/\sqrt{2.3.5}$
1.1	-3.1	2.2	−*	$+1/2.3\sqrt{5}$
1.1	0.2	-1.1	−	$-i\sqrt{5}/2.3\sqrt{2}$
-5.1	3.1	2.2	−	$+\sqrt{2}/\sqrt{3.5}$
-2.2	0.0	2.2	−	$+1/\sqrt{2.5}$
-2.2	3.1	-1.1	−	$-1/\sqrt{2.3.5}$
-2.2	0.2	2.2	+	$+i/\sqrt{3.5}$
4.2	-3.1	-1.1	−*	$+1/\sqrt{3.5}$
4.2	0.2	-4.0	−	$+i\sqrt{2}/\sqrt{3.5}$
1.3	-3.1	2.2	+*	$-2\sqrt{2}/3\sqrt{5}$
1.3	0.2	-1.1	+	$+2i/3\sqrt{5}$
31	**3**	**2**	**0**	
-2.0	0.2	2.2	+	$+1/\sqrt{3.5}$
1.1	-3.1	2.2	+	$+2/3\sqrt{5}$
1.1	0.2	-1.1	+	$-\sqrt{2}/3\sqrt{5}$
-5.1	3.3	2.2	−	$+\sqrt{2}/\sqrt{3.5}$
-2.2	0.2	2.2	−	$-1/\sqrt{3.5}$
-2.2	3.3	-1.1	−	$-\sqrt{2}/\sqrt{3.5}$
4.2	-6.0	2.2	+	$+1/\sqrt{2.5}$
4.2	-3.1	-1.1	+	$+1/\sqrt{3.5}$
4.2	0.2	-4.0	+	$+1/\sqrt{2.3.5}$
1.3	-3.1	2.2	−	$+\sqrt{2}/3\sqrt{5}$
1.3	0.2	-1.1	−	$-2/3\sqrt{5}$
1.3	3.3	-4.0	−	$+\sqrt{2}/\sqrt{3.5}$
31	**3^2**	**1^2**	**0**	
-2.0	3.1	-1.1	+	$+1/\sqrt{3.5}$
1.1	0.2	-1.1	−*	$+\sqrt{2}/\sqrt{3.5}$
-5.1	6.0	-1.1	+*	$-1/\sqrt{2.5}$
-5.1	3.1	2.0	+*	$+1/\sqrt{2.3.5}$
-2.2	3.1	-1.1	−	$+1/\sqrt{2.5}$
-2.2	0.2	2.0	−	$+1/\sqrt{2.5}$
4.2	-3.3	-1.1	+	$+1/\sqrt{5}$
1.3	0.2	-1.1	+*	$-1/\sqrt{3.5}$
1.3	-3.3	2.0	+*	$+1/\sqrt{5}$
31	**3^2**	**2**	**0**	
-2.0	6.0	-4.0	+	$+1/\sqrt{2.3.5}$
-2.0	3.1	-1.1	−*	0
-2.0	0.2	2.2	+	$-1/\sqrt{2.3.5}$

31	**3^2**	**2**	**0**	
1.1	3.1	-4.0	−*	$-1/\sqrt{3.5}$
1.1	0.2	-1.1	+	$+1/3\sqrt{5}$
1.1	-3.3	2.2	−*	$+\sqrt{2}/3\sqrt{5}$
-5.1	6.0	-1.1	−	$+1/\sqrt{3.5}$
-5.1	3.1	2.2	+*	$-1/\sqrt{3.5}$
-2.2	3.1	-1.1	+*	$-1/\sqrt{3.5}$
-2.2	0.2	2.2	−	$-\sqrt{2}/\sqrt{3.5}$
4.2	0.2	-4.0	+	$+1/\sqrt{3.5}$
4.2	-3.3	-1.1	−*	$+\sqrt{2}/\sqrt{3.5}$
1.3	0.2	-1.1	−	$+\sqrt{2}/3\sqrt{5}$
1.3	-3.3	2.2	+*	$-\sqrt{2}/3$
31	**31**	**1**	**0**	
1.1	-2.0	1.1	+	$-1/\sqrt{3.5}$
1.1	1.1	-2.0	+	$-1/3\sqrt{5}$
-2.2	1.1	1.1	−	$+\sqrt{2}/3\sqrt{5}$
4.2	-5.1	1.1	−	$+\sqrt{2}/\sqrt{3.5}$
4.2	-2.2	-2.0	−	$+1/\sqrt{3.5}$
1.3	-2.2	1.1	+	$-2/3\sqrt{5}$
1.3	1.3	-2.0	+	$+2\sqrt{2}/3\sqrt{5}$
31	**31**	**2^2**	**0**	
-2.0	-2.0	4.0	+	$+1/\sqrt{2.3.5}$
1.1	-2.0	1.1	−*	$-1/3\sqrt{2.5}$
1.1	1.1	-2.2	+	$+7/9\sqrt{2.5}$
-5.1	1.1	4.0	−	$-\sqrt{2}/3\sqrt{5}$
-2.2	1.1	1.1	+*	$+1/3\sqrt{3.5}$
-2.2	-2.2	4.0	+	$-\sqrt{2}/3\sqrt{5}$
4.2	-2.0	-2.2	+	$-1/3\sqrt{5}$
4.2	-5.1	1.1	+*	$+2/3\sqrt{5}$
4.2	-2.2	-2.2	−	$-2/3\sqrt{5}$
1.3	1.1	-2.2	−	$+2/9\sqrt{5}$
1.3	-2.2	1.1	−*	$-2\sqrt{2}/3\sqrt{3.5}$
1.3	1.3	-2.2	+	$-4/9$
31	**31**	**31**	**0**	
1.1	1.1	-2.0	+	$-\sqrt{2}/3\sqrt{3.5}$
-2.2	1.1	1.1	−	0
4.2	-5.1	1.1	−	$+1/3\sqrt{3.5}$
4.2	-2.2	-2.0	−	$+1/3\sqrt{2.5}$
4.2	-2.2	-2.2	+	$+\sqrt{2}/3\sqrt{5}$
1.3	-2.2	1.1	+	$+\sqrt{2}/3\sqrt{5}$
1.3	4.2	-5.1	+	$-2\sqrt{2}/3\sqrt{3.5}$
1.3	1.3	-2.0	+	$-2/3\sqrt{3.5}$
1.3	1.3	-2.2	−	0
31	**31**	**31**	**1**	
1.1	1.1	-2.0	−	0
-2.2	1.1	1.1	+	$+2/3\sqrt{3.5}$
4.2	-5.1	1.1	+	$-1/3\sqrt{5}$
4.2	-2.2	-2.0	+	$-1/\sqrt{2.3.5}$
4.2	-2.2	-2.2	−	0
1.3	-2.2	1.1	−	$-\sqrt{2}/3\sqrt{3.5}$
1.3	4.2	-5.1	−	$-\sqrt{2}/3\sqrt{5}$
1.3	1.3	-2.0	−	0
1.3	1.3	-2.2	+	$+\sqrt{2}/3\sqrt{3}$
32	**31**	**0**	**0**	
2.0	-2.0	0.0	+	$+1/\sqrt{3.5}$
-1.1	1.1	0.0	−*	$-\sqrt{2}/\sqrt{3.5}$

32	**31**	**0**	**0**	
5.1	-5.1	0.0	−*	$-\sqrt{2}/\sqrt{3.5}$
2.2	-2.2	0.0	+	$+1/\sqrt{5}$
-4.2	4.2	0.0	+	$+1/\sqrt{5}$
-1.3	1.3	0.0	−*	$-2/\sqrt{3.5}$
32	**31**	**21**	**0**	
2.0	-2.0	0.0	+	$+1/2\sqrt{3.5}$
2.0	1.1	-3.1	−*	$+1/2\sqrt{3.5}$
2.0	-5.1	3.1	−	0
2.0	-2.2	0.2	+	$+i/\sqrt{2.3.5}$
-1.1	1.1	0.0	−*	$-1/2\sqrt{2.3.5}$
-1.1	1.1	0.2	+*	$+7i/2.3\sqrt{2.3.5}$
-1.1	-2.2	3.1	+*	$+1/3\sqrt{2.5}$
-1.1	4.2	-3.1	+	$+1/3\sqrt{5}$
-1.1	1.3	0.2	−*	$+2i/3\sqrt{3.5}$
5.1	-5.1	0.0	−*	$-1/\sqrt{2.3.5}$
5.1	-5.1	0.2	+*	$-i/\sqrt{2.3.5}$
5.1	-2.2	-3.1	+	$-1/\sqrt{3.5}$
2.2	-2.2	0.0	+	0
2.2	-2.2	0.2	−	0
2.2	1.3	-3.1	−*	$-2/3\sqrt{5}$
-4.2	4.2	0.0	+	0
-4.2	4.2	0.2	−	$-i\sqrt{2}/\sqrt{3.5}$
-4.2	1.3	3.1	−	$-\sqrt{2}/3\sqrt{5}$
-1.3	1.3	0.0	−*	$+1/\sqrt{3.5}$
-1.3	1.3	0.2	+*	$+i/3\sqrt{3}$
32	**31**	**21**	**1**	
2.0	-2.0	0.0	−	0
2.0	1.1	-3.1	+*	$+i/3\sqrt{5}$
2.0	-5.1	3.1	+	$+i/\sqrt{2.3.5}$
2.0	-2.2	0.2	−	$+1/3\sqrt{2.5}$
-1.1	1.1	0.0	+*	$+i/3\sqrt{2.5}$
-1.1	1.1	0.2	−*	$-1/9\sqrt{2.5}$
-1.1	-2.2	3.1	−*	$-2i\sqrt{2}/3\sqrt{3.5}$
-1.1	4.2	-3.1	−	$+2i/3\sqrt{3.5}$
-1.1	1.3	0.2	+*	$+2/9\sqrt{5}$
5.1	-5.1	0.0	+*	$-i/2\sqrt{2.5}$
5.1	-5.1	0.2	−*	$-\sqrt{5}/2.3\sqrt{2}$
5.1	-2.2	-3.1	−	$-i/2.3\sqrt{5}$
2.2	-2.2	0.0	−	$+i/2\sqrt{3.5}$
2.2	-2.2	0.2	+	$+1/\sqrt{2.5}$
2.2	1.3	-3.1	+*	$-i/3\sqrt{3.5}$
-4.2	4.2	0.0	−	$-i/\sqrt{3.5}$
-4.2	4.2	0.2	+	$+1/3\sqrt{2.5}$
-4.2	1.3	3.1	+	$+i\sqrt{5}/3\sqrt{2.3}$
-1.3	1.3	0.0	+*	$-i/2.3\sqrt{5}$
-1.3	1.3	0.2	−*	$-7/2.9$
32	**31**	**3**	**0**	
2.0	1.1	-3.1	+	$-1/3\sqrt{5}$
2.0	-2.2	0.2	−	$+\sqrt{2}/3\sqrt{5}$
-1.1	1.1	0.2	−*	$-\sqrt{2}/9\sqrt{5}$
-1.1	-2.2	3.3	+*	$-2\sqrt{2}/3\sqrt{3.5}$
-1.1	4.2	-3.1	−*	$-2/3\sqrt{3.5}$
-1.1	1.3	0.2	+*	$+4/9\sqrt{5}$
5.1	-2.0	-3.1	+*	$+1/\sqrt{2.3.5}$
5.1	1.1	-6.0	+*	$-1/\sqrt{2.3.5}$

Table 10—*continued*

32	**31**	**3**	**0**			**32**	**31**	**3**	**0**			**32**	**31**	**3**	**0**		
5.1	-5.1	0.2		−*	$-\sqrt{2}/3\sqrt{5}$	2.2	4.2	-6.0		−	$-1/\sqrt{3.5}$	-1.3	1.1	0.2		+*	$+\sqrt{5}/9$
5.1	-2.2	-3.1		−*	$-1/3\sqrt{5}$	2.2	1.3	-3.1		+	$-2/3\sqrt{3.5}$	-1.3	-2.2	3.3		−*	$+\sqrt{2}/3\sqrt{3}$
2.2	-2.0	0.2		−	$+1/3\sqrt{2.5}$	-4.2	1.1	3.3		+	$+1/3\sqrt{3.5}$	-1.3	4.2	-3.1		+*	$-2\sqrt{2}/3\sqrt{3.5}$
2.2	1.1	-3.1		−	$-1/3\sqrt{2.3.5}$	-4.2	4.2	0.2		+	$+\sqrt{2}/3\sqrt{5}$	-1.3	1.3	0.2		−*	$+2/9$
2.2	-5.1	3.3		+	$+2/3\sqrt{5}$	-4.2	1.3	3.3		−	$+2/3\sqrt{3}$						
2.2	-2.2	0.2		+	0	-1.3	-2.0	3.3		+*	$-1/3\sqrt{5}$						

Table 11. Branching rules for $SU_6 \supset SU_2 \times SU_3$.

0	→	+ 0.0					
1	→	+ 1.1					
1^5	→	+ 1.1^2					
1^2	→	+ 0.2	+ 2.1^2				
1^4	→	+ 0.2^2	+ 2.1				
2	→	+ 0.1^2	+ 2.2				
2^5	→	+ 0.1	+ 2.2^2				
21^4	→	+ 0.21	+ 2.0	+ 2.21			
1^3	→	+ 1.21	+ 3.0				
21	→	+ 1.0	+ 1.21	+ 1.3	+ 3.21		
2^41	→	+ 1.0	+ 1.21	+ 1.3^2	+ 3.21		
21^3	→	+ 1.1^2	+ 1.2	+ 1.32	+ 3.1^2	+ 3.2	
2^21^3	→	+ 1.1	+ 1.2^2	+ 1.31	+ 3.1	+ 3.2^2	
3	→	+ 1.21	+ 3.3				
3^5	→	+ 1.21	+ 3.3^2				
31^4	→	+ 1.1	+ 1.2^2	+ 1.31	+ 3.1	+ 3.31	
32^4	→	+ 1.1^2	+ 1.2	+ 1.32	+ 3.1^2	+ 3.32	

Table 12. $3jm$ factors for $SU_6 \supset SU_2 \times SU_3$.

0	**0**	**0**	**0**			**2**	$\mathbf{1^5}$	$\mathbf{1^5}$	**0**		
0.0	0.0	0.0	0	+	$+1$	2.2	1.1^2	1.1^2	0	+	$+\sqrt{2.3}/\sqrt{7}$
$\mathbf{1^5}$	**1**	**0**	**0**			$\mathbf{2^5}$	**2**	**0**	**0**		
1.1^2	1.1	0.0	0	+	$+1$	0.1	0.1^2	0.0	0	+	$+1/\sqrt{7}$
$\mathbf{1^2}$	$\mathbf{1^5}$	$\mathbf{1^5}$	**0**			2.2^2	2.2	0.0	0	+	$+\sqrt{2.3}/\sqrt{7}$
0.2	1.1^2	1.1^2	0	+	$+\sqrt{2}/\sqrt{5}$	$\mathbf{21^4}$	$\mathbf{1^5}$	**1**	**0**		
2.1^2	1.1^2	1.1^2	0	+	$+\sqrt{3}/\sqrt{5}$	0.21	1.1^2	1.1	0	−	$-2i\sqrt{2}/\sqrt{5.7}$
$\mathbf{1^2}$	$\mathbf{1^2}$	$\mathbf{1^2}$	**0**			2.0	1.1^2	1.1	0	+	$+\sqrt{3}/\sqrt{5.7}$
0.2	0.2	0.2	0	+	$-1/\sqrt{5}$	2.21	1.1^2	1.1	0	+	$+2\sqrt{2.3}/\sqrt{5.7}$
2.1^2	2.1^2	0.2	0	+	$+1/\sqrt{5}$	$\mathbf{21^4}$	$\mathbf{1^4}$	$\mathbf{1^2}$	**0**		
2.1^2	2.1^2	2.1^2	0	+	$-1/\sqrt{5}$	0.21	0.2^2	0.2	0	−	$-i/\sqrt{7}$
$\mathbf{1^4}$	$\mathbf{1^2}$	**0**	**0**			0.21	2.1	2.1^2	0	−	$+i\sqrt{3}/\sqrt{5.7}$
0.2^2	0.2	0.0	0	+	$+\sqrt{2}/\sqrt{5}$	2.0	2.1	2.1^2	0	+	$+\sqrt{3}/\sqrt{5.7}$
2.1	2.1^2	0.0	0	+	$+\sqrt{3}/\sqrt{5}$	2.21	0.2^2	2.1^2	0	+	$+3/\sqrt{5.7}$
2	$\mathbf{1^5}$	$\mathbf{1^5}$	**0**			2.21	2.1	2.1^2	0	+	$-\sqrt{2.3}/\sqrt{5.7}$
0.1^2	1.1^2	1.1^2	0	+	$+1/\sqrt{7}$						

Table 12—*continued*

$\mathbf{21^4}$	$\mathbf{2}$	$\mathbf{1^4}$	$\mathbf{0}$		
0.21	0.1^2	0.2^2	0	−	$+i\sqrt{2}/\sqrt{5.7}$
0.21	2.2	2.1	0	−	$-i\sqrt{2.3}/\sqrt{5.7}$
2.0	0.1^2	2.1	0	+	$-1/\sqrt{5.7}$
2.0	2.2	0.2^2	0	+	$+\sqrt{2}/\sqrt{5.7}$
2.21	0.1^2	2.1	0	+	$+\sqrt{2}/\sqrt{5.7}$
2.21	2.2	0.2^2	0	+	$+\sqrt{2}/\sqrt{7}$
2.21	2.2	2.1	0	+	$-2\sqrt{3}/\sqrt{5.7}$
$\mathbf{21^4}$	$\mathbf{2^5}$	$\mathbf{2}$	$\mathbf{0}$		
0.21	0.1	0.1^2	0	−	$-i/\sqrt{2.5.7}$
0.21	2.2^2	2.2	0	−	$+i\sqrt{3}/\sqrt{2.7}$
2.0	2.2^2	2.2	0	+	$-\sqrt{3}/\sqrt{5.7}$
2.21	0.1	2.2	0	+	$+3/\sqrt{2.5.7}$
2.21	2.2^2	2.2	0	+	$-\sqrt{3}/\sqrt{7}$
$\mathbf{21^4}$	$\mathbf{21^4}$	$\mathbf{0}$	$\mathbf{0}$		
0.21	0.21	0.0	0	+	$+2\sqrt{2}/\sqrt{5.7}$
2.0	2.0	0.0	0	+	$+\sqrt{3}/\sqrt{5.7}$
2.21	2.21	0.0	0	+	$+2\sqrt{2.3}/\sqrt{5.7}$
$\mathbf{21^4}$	$\mathbf{21^4}$	$\mathbf{21^4}$	$\mathbf{0}$		
0.21	0.21	0.21	0	−	0
0.21	0.21	0.21	1	+	$+\sqrt{2}/\sqrt{5.7}$
2.0	2.0	2.0	0	+	$-1/\sqrt{3.5.7}$
2.21	2.0	0.21	0	−	0
2.21	2.21	0.21	0	−	0
2.21	2.21	0.21	1	+	$+\sqrt{2.3}/\sqrt{5.7}$
2.21	2.21	2.0	0	+	$-2\sqrt{2}/\sqrt{3.5.7}$
2.21	2.21	2.21	0	+	$-2/\sqrt{3.7}$
2.21	2.21	2.21	1	−	0
$\mathbf{21^4}$	$\mathbf{21^4}$	$\mathbf{21^4}$	$\mathbf{1}$		
0.21	0.21	0.21	0	+	$-1/2\sqrt{7}$
0.21	0.21	0.21	1	−	0
2.0	2.0	2.0	0	−	0
2.21	2.0	0.21	0	+	$-\sqrt{3}/\sqrt{2.5.7}$
2.21	2.21	0.21	0	+	$-\sqrt{3}/2\sqrt{7}$
2.21	2.21	0.21	1	−	0
2.21	2.21	2.0	0	−	0
2.21	2.21	2.21	0	−	0
2.21	2.21	2.21	1	+	$-3\sqrt{3}/\sqrt{2.5.7}$
$\mathbf{1^3}$	$\mathbf{1^2}$	$\mathbf{1}$	$\mathbf{0}$		
1.21	0.2	1.1	0	+	$+\sqrt{2}/\sqrt{5}$
1.21	2.1^2	1.1	0	+	$+\sqrt{2}/\sqrt{5}$
3.0	2.1^2	1.1	0	−	$-1/\sqrt{5}$
$\mathbf{1^3}$	$\mathbf{1^3}$	$\mathbf{0}$	$\mathbf{0}$		
1.21	1.21	0.0	0	+	$+2/\sqrt{5}$
3.0	3.0	0.0	0	+	$+1/\sqrt{5}$
$\mathbf{1^3}$	$\mathbf{1^3}$	$\mathbf{21^4}$	$\mathbf{0}$		
1.21	1.21	0.21	0	−	0
1.21	1.21	0.21	1	+	$+2\sqrt{2}/\sqrt{5.7}$
1.21	1.21	2.0	0	+	$-2/\sqrt{3.5.7}$
1.21	1.21	2.21	0	+	$+2\sqrt{2}/\sqrt{3.7}$
1.21	1.21	2.21	1	−	0
3.0	1.21	2.21	0	−	$-4/\sqrt{3.5.7}$
3.0	3.0	2.0	0	+	$-1/\sqrt{3.7}$
$\mathbf{21}$	$\mathbf{1^4}$	$\mathbf{1^5}$	$\mathbf{0}$		
1.0	2.1	1.1^2	0	+	$+1/\sqrt{5.7}$
1.21	0.2^2	1.1^2	0	+	$-2/\sqrt{5.7}$
$\mathbf{21}$	$\mathbf{1^4}$	$\mathbf{1^5}$	$\mathbf{0}$		
1.21	2.1	1.1^2	0	+	$+2/\sqrt{5.7}$
1.3	0.2^2	1.1^2	0	−	$-\sqrt{2}/\sqrt{7}$
3.21	2.1	1.1^2	0	−	$-4/\sqrt{5.7}$
$\mathbf{21}$	$\mathbf{2^5}$	$\mathbf{1^5}$	$\mathbf{0}$		
1.0	0.1	1.1^2	0	+	$-1/\sqrt{5.7}$
1.21	0.1	1.1^2	0	+	$+2/\sqrt{5.7}$
1.21	2.2^2	1.1^2	0	+	$+2/\sqrt{5.7}$
1.3	2.2^2	1.1^2	0	−	$-\sqrt{2}/\sqrt{7}$
3.21	2.2^2	1.1^2	0	−	$+4/\sqrt{5.7}$
$\mathbf{21}$	$\mathbf{1^3}$	$\mathbf{21^4}$	$\mathbf{0}$		
1.0	1.21	0.21	0	−	$-i/\sqrt{2.5.7}$
1.0	1.21	2.21	0	+	$+1/\sqrt{2.3.5.7}$
1.0	3.0	2.0	0	−	$+1/\sqrt{3.5.7}$
1.21	1.21	0.21	0	−	$-i/\sqrt{2.7}$
1.21	1.21	0.21	1	+	$+1/\sqrt{2.5.7}$
1.21	1.21	2.0	0	+	$+2/\sqrt{3.5.7}$
1.21	1.21	2.21	0	+	$-1/\sqrt{2.3.7}$
1.21	1.21	2.21	1	−	$+i\sqrt{3}/\sqrt{2.5.7}$
1.21	3.0	2.21	0	−	$+2/\sqrt{3.5.7}$
1.3	1.21	0.21	0	+	$+i/\sqrt{2.7}$
1.3	1.21	2.21	0	−	$+\sqrt{3}/\sqrt{2.7}$
3.21	1.21	2.0	0	−	$-2/\sqrt{3.5.7}$
3.21	1.21	2.21	0	−	$-\sqrt{2}/\sqrt{3.7}$
3.21	1.21	2.21	1	+	$-i\sqrt{2.3}/\sqrt{5.7}$
3.21	3.0	0.21	0	−	$-i\sqrt{2}/\sqrt{5.7}$
3.21	3.0	2.21	0	+	$-\sqrt{2}/\sqrt{3.7}$
$\mathbf{2^41}$	$\mathbf{21}$	$\mathbf{0}$	$\mathbf{0}$		
1.0	1.0	0.0	0	+	$+1/\sqrt{5.7}$
1.21	1.21	0.0	0	+	$+2\sqrt{2}/\sqrt{5.7}$
1.3^2	1.3	0.0	0	+	$+\sqrt{2}/\sqrt{7}$
3.21	3.21	0.0	0	+	$+4/\sqrt{5.7}$
$\mathbf{2^41}$	$\mathbf{21}$	$\mathbf{21^4}$	$\mathbf{0}$		
1.0	1.0	2.0	0	+	$+1/9\sqrt{5.7}$
1.0	1.21	0.21	0	−	$+2i/3\sqrt{3.5.7}$
1.0	1.21	2.21	0	+	$+2/9\sqrt{5.7}$
1.0	3.21	2.21	0	−	$+8/9\sqrt{5.7}$
1.21	1.21	0.21	0	−	0
1.21	1.21	0.21	1	+	$+4/3\sqrt{3.5.7}$
1.21	1.21	2.0	0	+	$-2\sqrt{2}/9\sqrt{5.7}$
1.21	1.21	2.21	0	+	$+4/9\sqrt{7}$
1.21	1.21	2.21	1	−	0
1.21	1.3	0.21	0	+	$-2i/3\sqrt{3.7}$
1.21	1.3	2.21	0	−	$+2/3\sqrt{7}$
1.21	3.21	2.0	0	−	$+4\sqrt{2}/9\sqrt{5.7}$
1.21	3.21	2.21	0	−	$+4/9\sqrt{7}$
1.21	3.21	2.21	1	+	$-4i/3\sqrt{5.7}$
1.3^2	1.3	0.21	0	−	$+2i\sqrt{2}/3\sqrt{3.7}$
1.3^2	1.3	2.0	0	+	$-\sqrt{2}/3\sqrt{7}$
1.3^2	1.3	2.21	0	+	$-2\sqrt{2}/3\sqrt{7}$
1.3^2	3.21	2.21	0	+	0
3.21	3.21	0.21	0	−	$+2i\sqrt{2}/3\sqrt{3.7}$
3.21	3.21	0.21	1	+	$+2\sqrt{2}/\sqrt{3.5.7}$
3.21	3.21	2.0	0	+	$-4/9\sqrt{7}$
3.21	3.21	2.21	0	+	$-2\sqrt{2.5}/9\sqrt{7}$
3.21	3.21	2.21	1	−	$+2i\sqrt{2}/3\sqrt{7}$

Table 12—*continued*

2^41	**21**	**21^4**	**1**		
1.0	1.0	2.0	0	−	$-i\sqrt{5}/9\sqrt{2.7}$
1.0	1.21	0.21	0	+	$+13/4.3\sqrt{2.3.5.7}$
1.0	1.21	2.21	0	−	$+41i/4.9\sqrt{2.5.7}$
1.0	3.21	2.21	0	+	$+i/2.9\sqrt{2.5.7}$
1.21	1.21	0.21	0	+	0
1.21	1.21	0.21	1	−	$+i\sqrt{7}/3\sqrt{2.3.5}$
1.21	1.21	2.0	0	−	$-i\sqrt{7}/2.9\sqrt{5}$
1.21	1.21	2.21	0	−	$-13i/2.9\sqrt{2.7}$
1.21	1.21	2.21	1	+	$-3/2\sqrt{2.5.7}$
1.21	1.3	0.21	0	−	$-13/4.3\sqrt{2.3.7}$
1.21	1.3	2.21	0	+	$+5i/4.3\sqrt{2.7}$
1.21	3.21	2.0	0	+	$-13i/2.9\sqrt{5.7}$
1.21	3.21	2.21	0	+	$+i\sqrt{7}/9\sqrt{2}$
1.21	3.21	2.21	1	−	$-\sqrt{2}/3\sqrt{5.7}$
1.3^2	1.3	0.21	0	+	$-\sqrt{7}/2.3\sqrt{3}$
1.3^2	1.3	2.0	0	−	$+i/2.3\sqrt{7}$
1.3^2	1.3	2.21	0	−	$+i/3\sqrt{7}$
1.3^2	3.21	2.21	0	−	$+3i/2\sqrt{2.7}$
3.21	3.21	0.21	0	+	$+13/4.3\sqrt{3.7}$
3.21	3.21	0.21	1	−	$+i\sqrt{5}/4\sqrt{3.7}$
3.21	3.21	2.0	0	−	$-i\sqrt{7}/9\sqrt{2}$
3.21	3.21	2.21	0	−	$+13i\sqrt{5}/4.9\sqrt{7}$
3.21	3.21	2.21	1	+	$-5/4.3\sqrt{7}$
21^3	**1^2**	**1^5**	**0**		
1.1^2	0.2	1.1^2	0	−	$+\sqrt{3}/\sqrt{2.5.7}$
1.1^2	2.1^2	1.1^2	0	−	$-1/\sqrt{5.7}$
1.2	2.1^2	1.1^2	0	+	$+1/\sqrt{7}$
1.32	0.2	1.1^2	0	−	$-\sqrt{5}/\sqrt{2.7}$
3.1^2	2.1^2	1.1^2	0	+	$+1/\sqrt{7}$
3.2	2.1^2	1.1^2	0	−	$-\sqrt{2}/\sqrt{7}$
21^3	**21^4**	**1**	**0**		
1.1^2	0.21	1.1	0	+	$-3i/2\sqrt{2.5.7}$
1.1^2	2.0	1.1	0	−	$+2/\sqrt{3.5.7}$
1.1^2	2.21	1.1	0	−	$+1/2\sqrt{2.3.5.7}$
1.2	0.21	1.1	0	−	$+i\sqrt{3}/2\sqrt{7}$
1.2	2.21	1.1	0	+	$+1/2\sqrt{7}$
1.32	0.21	1.1	0	+	$-i\sqrt{5}/2\sqrt{2.7}$
1.32	2.21	1.1	0	−	$+\sqrt{3.5}/2\sqrt{2.7}$
3.1^2	2.0	1.1	0	+	$+1/\sqrt{3.7}$
3.1^2	2.21	1.1	0	+	$-\sqrt{2}/\sqrt{3.7}$
3.2	2.21	1.1	0	−	$+\sqrt{2}/\sqrt{7}$
21^3	**1^3**	**1^4**	**0**		
1.1^2	1.21	0.2^2	0	−	$+1/2\sqrt{5.7}$
1.1^2	1.21	2.1	0	−	$-\sqrt{7}/2.3\sqrt{5}$
1.1^2	3.0	2.1	0	+	$-2\sqrt{2}/3\sqrt{5.7}$
1.2	1.21	0.2^2	0	+	$+\sqrt{5}/\sqrt{2.3.7}$
1.2	1.21	2.1	0	+	$+1/\sqrt{2.3.7}$
1.32	1.21	0.2^2	0	−	$+\sqrt{5}/2\sqrt{7}$
1.32	1.21	2.1	0	−	$+\sqrt{5}/2\sqrt{7}$
3.1^2	1.21	2.1	0	+	$-2/3\sqrt{7}$
3.1^2	3.0	2.1	0	−	$+\sqrt{5}/3\sqrt{7}$
3.2	1.21	2.1	0	−	$-2/\sqrt{3.7}$
3.2	3.0	0.2^2	0	+	$+\sqrt{2}/\sqrt{3.7}$
21^3	**1^3**	**2^5**	**0**		
1.1^2	1.21	0.1	0	−	$+1/2\sqrt{7}$
1.1^2	1.21	2.2^2	0	−	$+1/2\sqrt{7}$
1.2	1.21	0.1	0	+	$+\sqrt{3}/\sqrt{2.5.7}$
1.2	1.21	2.2^2	0	+	$+1/\sqrt{2.3.7}$
1.2	3.0	2.2^2	0	−	$+2\sqrt{2}/\sqrt{3.5.7}$
1.32	1.21	0.1	0	−	$-1/2\sqrt{7}$
1.32	1.21	2.2^2	0	−	$-3/2\sqrt{7}$
3.1^2	1.21	2.2^2	0	+	$-2/\sqrt{5.7}$
3.1^2	3.0	0.1	0	−	$-1/\sqrt{5.7}$
3.2	1.21	2.2^2	0	−	$-2/\sqrt{3.7}$
3.2	3.0	2.2^2	0	+	$-\sqrt{2}/\sqrt{3.7}$
21^3	**21**	**1^4**	**0**		
1.1^2	1.0	2.1	0	−	$+1/2.3\sqrt{5.7}$
1.1^2	1.21	0.2^2	0	−	$-3/4\sqrt{5.7}$
1.1^2	1.21	2.1	0	−	$-11/4.3\sqrt{5.7}$
1.1^2	1.3	0.2^2	0	+	$+1/2\sqrt{2.7}$
1.1^2	3.21	2.1	0	+	$-2/3\sqrt{5.7}$
1.2	1.0	0.2^2	0	+	$-1/2\sqrt{3.7}$
1.2	1.21	0.2^2	0	+	$-\sqrt{5}/2\sqrt{2.3.7}$
1.2	1.21	2.1	0	+	$-1/2\sqrt{2.3.7}$
1.2	3.21	2.1	0	−	$-\sqrt{2}/\sqrt{3.7}$
1.32	1.21	0.2^2	0	−	$+\sqrt{5}/4\sqrt{7}$
1.32	1.21	2.1	0	−	$+\sqrt{5}/4\sqrt{7}$
1.32	1.3	0.2^2	0	+	$-\sqrt{5}/2\sqrt{2.7}$
1.32	1.3	2.1	0	+	$+\sqrt{5}/2\sqrt{7}$
1.32	3.21	2.1	0	+	0
3.1^2	1.0	2.1	0	+	$-1/3\sqrt{7}$
3.1^2	1.21	2.1	0	+	$+1/3\sqrt{7}$
3.1^2	3.21	0.2^2	0	−	$-1/\sqrt{2.7}$
3.1^2	3.21	2.1	0	−	$+\sqrt{5}/3\sqrt{2.7}$
3.2	1.21	2.1	0	−	$+1/\sqrt{3.7}$
3.2	3.21	0.2^2	0	+	$-\sqrt{5}/\sqrt{2.3.7}$
3.2	3.21	2.1	0	+	$+\sqrt{5}/\sqrt{2.3.7}$
21^3	**21**	**2^5**	**0**		
1.1^2	1.0	0.1	0	−	$-1/4\sqrt{5.7}$
1.1^2	1.21	0.1	0	−	$-1/8\sqrt{5.7}$
1.1^2	1.21	2.2^2	0	−	$-1/8\sqrt{5.7}$
1.1^2	1.3	2.2^2	0	+	$+3/4\sqrt{2.7}$
1.1^2	3.21	2.2^2	0	+	$+1/\sqrt{5.7}$
1.2	1.0	2.2^2	0	+	$+1/4\sqrt{3.7}$
1.2	1.21	0.1	0	+	$+\sqrt{3}/4\sqrt{2.7}$
1.2	1.21	2.2^2	0	+	$+\sqrt{5}/4\sqrt{2.3.7}$
1.2	3.21	2.2^2	0	−	$+\sqrt{5}/\sqrt{2.3.7}$
1.32	1.21	0.1	0	−	$+\sqrt{5}/8\sqrt{7}$
1.32	1.21	2.2^2	0	−	$+3\sqrt{5}/8\sqrt{7}$
1.32	1.3	0.1	0	+	$+\sqrt{5}/4\sqrt{7}$
1.32	1.3	2.2^2	0	+	$-3\sqrt{5}/4\sqrt{2.7}$
1.32	3.21	2.2^2	0	+	0
3.1^2	1.21	2.2^2	0	+	$-1/2\sqrt{7}$
3.1^2	1.3	2.2^2	0	−	0
3.1^2	3.21	0.1	0	−	$+1/2\sqrt{2.7}$
3.1^2	3.21	2.2^2	0	−	$-\sqrt{5}/2\sqrt{2.7}$
3.2	1.0	2.2^2	0	−	$-1/\sqrt{2.3.7}$
3.2	1.21	2.2^2	0	−	$-\sqrt{5}/2\sqrt{3.7}$
3.2	3.21	0.1	0	+	$+\sqrt{3}/2\sqrt{2.7}$
3.2	3.21	2.2^2	0	+	$-5/2\sqrt{2.3.7}$

Table 12—*continued*

21^3	2^41	1^4	0		
1.1^2	1.0	2.1	0	−	$-1/3\sqrt{7}$
1.1^2	1.21	0.2^2	0	−	$+1/2\sqrt{7}$
1.1^2	1.21	2.1	0	−	$-1/2.3\sqrt{7}$
1.1^2	3.21	2.1	0	+	$+1/3\sqrt{7}$
1.2	1.0	0.2^2	0	+	$-1/\sqrt{3.5.7}$
1.2	1.21	0.2^2	0	+	$-1/\sqrt{2.3.7}$
1.2	1.21	2.1	0	+	$-\sqrt{3}/\sqrt{2.5.7}$
1.2	1.3^2	2.1	0	−	$-1/\sqrt{3.7}$
1.2	3.21	2.1	0	−	$-\sqrt{2}/\sqrt{3.5.7}$
1.32	1.21	0.2^2	0	−	$+1/2\sqrt{7}$
1.32	1.21	2.1	0	−	$-1/2\sqrt{7}$
1.32	1.3^2	0.2^2	0	+	$+1/\sqrt{7}$
1.32	3.21	2.1	0	+	$+1/\sqrt{7}$
3.1^2	1.0	2.1	0	+	$+1/3\sqrt{5.7}$
3.1^2	1.21	2.1	0	+	$-4/3\sqrt{5.7}$
3.1^2	3.21	0.2^2	0	−	$+\sqrt{2}/\sqrt{5.7}$
3.1^2	3.21	2.1	0	−	$+\sqrt{2}/3\sqrt{7}$
3.2	1.21	2.1	0	−	0
3.2	1.3^2	2.1	0	+	$-\sqrt{2}/\sqrt{3.7}$
3.2	3.21	0.2^2	0	+	$-\sqrt{2}/\sqrt{3.7}$
3.2	3.21	2.1	0	+	$-\sqrt{2}/\sqrt{3.7}$
21^3	21^3	1^2	0		
1.1^2	1.1^2	0.2	0	+	$+17/8.3\sqrt{5.7}$
1.1^2	1.1^2	2.1^2	0	+	$+1/4\sqrt{2.3.5.7}$
1.2	1.1^2	2.1^2	0	−	$+13/4.3\sqrt{2.3.7}$
1.2	1.2	0.2	0	+	$+\sqrt{5}/4\sqrt{3.7}$
1.32	1.1^2	0.2	0	+	$+\sqrt{5}/8\sqrt{3.7}$
1.32	1.2	0.2	0	−	$+5/4\sqrt{3.7}$
1.32	1.2	2.1^2	0	−	$-5/4.3\sqrt{2.7}$
1.32	1.32	0.2	0	+	$+5/8\sqrt{7}$
1.32	1.32	2.1^2	0	+	$+5/4\sqrt{2.7}$
3.1^2	1.1^2	2.1^2	0	−	$-1/\sqrt{2.3.7}$
3.1^2	1.2	2.1^2	0	+	$-\sqrt{5}/3\sqrt{2.3.7}$
3.1^2	3.1^2	0.2	0	+	$+\sqrt{5}/3\sqrt{2.7}$
3.1^2	3.1^2	2.1^2	0	+	0
3.2	1.1^2	2.1^2	0	+	$-1/2.3\sqrt{3.7}$
3.2	1.32	2.1^2	0	+	$-5/2.3\sqrt{7}$
3.2	3.1^2	2.1^2	0	−	$-5/3\sqrt{2.3.7}$
3.2	3.2	0.2	0	+	$-\sqrt{5}/\sqrt{2.3.7}$
21^3	21^3	2	0		
1.1^2	1.1^2	0.1^2	0	+	$+\sqrt{3}/4\sqrt{2.7}$
1.1^2	1.1^2	2.2	0	+	$-\sqrt{7}/8.3$
1.2	1.1^2	0.1^2	0	−	$+\sqrt{5}/4\sqrt{2.3.7}$
1.2	1.2	2.2	0	+	$+5/4\sqrt{3.7}$
1.32	1.1^2	2.2	0	+	$+5/8\sqrt{3.7}$
1.32	1.2	0.1^2	0	−	$-\sqrt{5}/4\sqrt{2.7}$
1.32	1.2	2.2	0	−	$+\sqrt{5}/4\sqrt{3.7}$
1.32	1.32	0.1^2	0	+	$-\sqrt{5}/4\sqrt{2.7}$
1.32	1.32	2.2	0	+	$-3\sqrt{5}/8\sqrt{7}$
3.1^2	1.1^2	2.2	0	−	$-\sqrt{5}/2.3\sqrt{7}$
3.1^2	1.32	2.2	0	−	$-\sqrt{5}/2\sqrt{3.7}$
3.1^2	3.1^2	0.1^2	0	+	0
3.1^2	3.1^2	2.2	0	+	$+\sqrt{5}/3\sqrt{2.7}$
3.2	1.2	2.2	0	−	$+1/\sqrt{2.3.7}$
3.2	1.32	2.2	0	+	$+\sqrt{5}/\sqrt{2.3.7}$
3.2	3.1^2	0.1^2	0	−	$-1/\sqrt{2.3.7}$
3.2	3.2	2.2	0	+	$-\sqrt{5}/\sqrt{2.3.7}$
2^21^3	21^3	0	0		
1.1	1.1^2	0.0	0	+	$+1/\sqrt{2.7}$
1.2^2	1.2	0.0	0	+	$+1/\sqrt{7}$
1.31	1.32	0.0	0	+	$+\sqrt{5}/\sqrt{2.7}$
3.1	3.1^2	0.0	0	+	$+1/\sqrt{7}$
3.2^2	3.2	0.0	0	+	$+\sqrt{2}/\sqrt{7}$
2^21^3	21^3	21^4	0		
1.1	1.1^2	0.21	0	−	$+31i/8.4\sqrt{3.5.7}$
1.1	1.1^2	2.0	0	+	$+13/8.9\sqrt{2.5.7}$
1.1	1.1^2	2.21	0	+	$-83/8.2.9\sqrt{5.7}$
1.1	1.2	0.21	0	+	$+i\sqrt{7}/8.3\sqrt{2}$
1.1	1.2	2.21	0	−	$+13/8.2\sqrt{2.3.7}$
1.1	1.32	0.21	0	−	$+i\sqrt{5.7}/8.4\sqrt{3}$
1.1	1.32	2.21	0	+	$+\sqrt{5}/8.3\sqrt{7}$
1.1	3.1^2	2.0	0	−	$+\sqrt{7}/4.9\sqrt{2}$
1.1	3.1^2	2.21	0	−	$+47/8.2.9\sqrt{7}$
1.1	3.2	2.21	0	+	$+1/8.2\sqrt{3.7}$
1.2^2	1.2	0.21	0	−	$-5i/8.2\sqrt{3.7}$
1.2^2	1.2	2.0	0	+	$+11\sqrt{5}/8.9\sqrt{7}$
1.2^2	1.2	2.21	0	+	$-5/2.9\sqrt{7}$
1.2^2	1.32	0.21	0	+	0
1.2^2	1.32	2.21	0	−	$+5.5/8.2\sqrt{2.3.7}$
1.2^2	3.1^2	2.21	0	+	$-\sqrt{3.5}/8\sqrt{2.7}$
1.2^2	3.2	2.0	0	−	$+\sqrt{5.7}/4.9\sqrt{2}$
1.2^2	3.2	2.21	0	−	$-5/8.9\sqrt{2.7}$
1.31	1.32	0.21	0	−	$+3.5i\sqrt{3}/8.4\sqrt{7}$
1.31	1.32	0.21	1	+	$-5/8.2\sqrt{7}$
1.31	1.32	2.0	0	+	$-5/8.3\sqrt{2.7}$
1.31	1.32	2.21	0	+	$-5/8.2.3\sqrt{7}$
1.31	1.32	2.21	1	−	$+5i/8\sqrt{3.7}$
1.31	3.1^2	2.21	0	−	$-5.5/8.2.3\sqrt{7}$
1.31	3.2	2.21	0	+	$+5.5/8.2\sqrt{3.7}$
3.1	3.1^2	0.21	0	−	$+i\sqrt{5}/8\sqrt{2.3.7}$
3.1	3.1^2	2.0	0	+	$+5/2.9\sqrt{7}$
3.1	3.1^2	2.21	0	+	$+5\sqrt{7}/8.9\sqrt{2}$
3.1	3.2	0.21	0	+	$+i\sqrt{5.7}/8.3\sqrt{2}$
3.1	3.2	2.21	0	−	$+5/8\sqrt{2.3.7}$
3.2^2	3.2	0.21	0	−	$-5i/8\sqrt{2.3.7}$
3.2^2	3.2	2.0	0	+	$+5/9\sqrt{2.7}$
3.2^2	3.2	2.21	0	+	$-5\sqrt{5.7}/8.9\sqrt{2}$
2^21^3	21^3	21^4	1		
1.1	1.1^2	0.21	0	+	$-\sqrt{5}/8.4\sqrt{7}$
1.1	1.1^2	2.0	0	−	$+i\sqrt{5}/8\sqrt{2.3.7}$
1.1	1.1^2	2.21	0	−	$+i\sqrt{5}/8.2\sqrt{3.7}$
1.1	1.2	0.21	0	−	$+\sqrt{3}/8\sqrt{2.7}$
1.1	1.2	2.21	0	+	$-i/8.2\sqrt{2.7}$
1.1	1.32	0.21	0	+	$+3\sqrt{5}/8.4\sqrt{7}$
1.1	1.32	2.21	0	−	$+i\sqrt{3.5}/8\sqrt{7}$
1.1	3.1^2	2.0	0	+	$-i/4\sqrt{2.3.7}$
1.1	3.1^2	2.21	0	+	$+i\sqrt{7}/8.2\sqrt{3}$
1.1	3.2	2.21	0	−	$-5i/8.2\sqrt{7}$
1.2^2	1.2	0.21	0	+	$+\sqrt{7}/8.2$
1.2^2	1.2	2.0	0	−	$-i/8\sqrt{3.5.7}$

Table 12—*continued*

2^21^3	**21^3**	**21^4**	**1**		
1.2^2	1.2	2.21	0	−	$-i/2\sqrt{3.7}$
1.2^2	1.32	0.21	0	−	0
1.2^2	1.32	2.21	0	+	$+3i/8.2\sqrt{2.7}$
1.2^2	3.1^2	2.21	0	−	$+11i/8\sqrt{2.5.7}$
1.2^2	3.2	2.0	0	+	$-i\sqrt{5}/4\sqrt{2.3.7}$
1.2^2	3.2	2.21	0	+	$-13i/8\sqrt{2.3.7}$
1.31	1.32	0.21	0	+	$+1/8.4\sqrt{7}$
1.31	1.32	0.21	1	−	$-i\sqrt{3.7}/8.2$
1.31	1.32	2.0	0	−	$+3i\sqrt{3}/8\sqrt{2.7}$
1.31	1.32	2.21	0	−	$+3i\sqrt{3}/8.2\sqrt{7}$
1.31	1.32	2.21	1	+	$+9/8\sqrt{7}$
1.31	3.1^2	2.21	0	+	$-i\sqrt{3}/8.2\sqrt{7}$
1.31	3.2	2.21	0	−	$+3i/8.2\sqrt{7}$
3.1	3.1^2	0.21	0	+	$-\sqrt{7}/8\sqrt{2.5}$
3.1	3.1^2	2.0	0	−	$+i/2\sqrt{3.7}$
3.1	3.1^2	2.21	0	−	$-5i/8\sqrt{2.3.7}$
3.1	3.2	0.21	0	−	$+\sqrt{3.5}/8\sqrt{2.7}$
3.1	3.2	2.21	0	+	$+i\sqrt{7}/8\sqrt{2}$
3.2^2	3.2	0.21	0	+	$+\sqrt{7}/8\sqrt{2}$
3.2^2	3.2	2.0	0	−	$+i/\sqrt{2.3.7}$
3.2^2	3.2	2.21	0	−	$+5i\sqrt{5}/8\sqrt{2.3.7}$
3	**2^5**	**1^5**	**0**		
1.21	0.1	1.1^2	0	+	$+1/\sqrt{7}$
1.21	2.2^2	1.1^2	0	+	$-1/\sqrt{7}$
3.3	2.2^2	1.1^2	0	+	$+\sqrt{5}/\sqrt{7}$
3	**2^41**	**21^4**	**0**		
1.21	1.0	0.21	0	−	$-i/2\sqrt{5.7}$
1.21	1.0	2.21	0	+	$-\sqrt{3}/2\sqrt{5.7}$
1.21	1.21	0.21	0	−	$+i/2\sqrt{7}$
1.21	1.21	0.21	1	+	$-1/2\sqrt{5.7}$
1.21	1.21	2.0	0	+	$+\sqrt{2}/\sqrt{3.5.7}$
1.21	1.21	2.21	0	+	$+1/2\sqrt{3.7}$
1.21	1.21	2.21	1	−	$+i\sqrt{5}/2\sqrt{3.7}$
1.21	1.3^2	0.21	0	+	$-i/2\sqrt{7}$
1.21	1.3^2	2.21	0	−	$+1/2\sqrt{3.7}$
1.21	3.21	2.0	0	−	$-\sqrt{2}/\sqrt{3.5.7}$
1.21	3.21	2.21	0	−	$+1/\sqrt{3.7}$
1.21	3.21	2.21	1	+	$-i/\sqrt{3.5.7}$
3.3	1.21	2.21	0	+	$-\sqrt{2}/\sqrt{3.7}$
3.3	1.3^2	2.0	0	−	$+1/\sqrt{3.7}$
3.3	1.3^2	2.21	0	−	$+2/\sqrt{3.7}$
3.3	3.21	0.21	0	+	$+i/\sqrt{7}$
3.3	3.21	2.21	0	−	$+\sqrt{5}/\sqrt{3.7}$
3^5	**3**	**0**	**0**		
1.21	1.21	0.0	0	+	$+\sqrt{2}/\sqrt{7}$
3.3^2	3.3	0.0	0	+	$+\sqrt{5}/\sqrt{7}$
3^5	**3**	**21^4**	**0**		
1.21	1.21	0.21	0	−	0
1.21	1.21	0.21	1	+	$-2/\sqrt{3.5.7}$
1.21	1.21	2.0	0	+	$+\sqrt{2}/3\sqrt{5.7}$
1.21	1.21	2.21	0	+	$+2/3\sqrt{7}$
1.21	1.21	2.21	1	−	$-4i/3\sqrt{5.7}$
1.21	3.3	2.21	0	+	$+2\sqrt{2}/3\sqrt{7}$
3.3^2	3.3	0.21	0	−	$-2i/\sqrt{3.7}$
3.3^2	3.3	2.0	0	+	$+\sqrt{5}/3\sqrt{7}$

3^5	**3**	**21^4**	**0**		
3.3^2	3.3	2.21	0	+	$+2\sqrt{5}/3\sqrt{7}$
31^4	**2^5**	**1**	**0**		
1.1	0.1	1.1	0	−	$+\sqrt{3}/\sqrt{2.5.7}$
1.1	2.2^2	1.1	0	−	$-1/2\sqrt{5.7}$
1.2^2	0.1	1.1	0	+	$+1/\sqrt{2.5}$
1.31	2.2^2	1.1	0	−	$-1/2$
3.1	2.2^2	1.1	0	+	$+1/\sqrt{2.5}$
3.31	2.2^2	1.1	0	+	$+1/\sqrt{2}$
31^4	**21^4**	**1^5**	**0**		
1.1	0.21	1.1^2	0	+	$+3i/4\sqrt{5.7}$
1.1	2.0	1.1^2	0	−	$-\sqrt{2}/\sqrt{3.5.7}$
1.1	2.21	1.1^2	0	−	$+\sqrt{5}/4\sqrt{3.7}$
1.2^2	0.21	1.1^2	0	−	$+i/2\sqrt{2.5}$
1.2^2	2.21	1.1^2	0	+	$+\sqrt{3}/2\sqrt{2.5}$
1.31	0.21	1.1^2	0	+	$-i\sqrt{3}/4$
1.31	2.21	1.1^2	0	−	$+1/4$
3.1	2.0	1.1^2	0	+	$-1/\sqrt{3.5}$
3.1	2.21	1.1^2	0	+	$-1/\sqrt{2.3.5}$
3.31	2.21	1.1^2	0	+	$+1/\sqrt{2}$
31^4	**2^41**	**1^2**	**0**		
1.1	1.0	2.1^2	0	−	$-3/4\sqrt{5.7}$
1.1	1.21	0.2	0	−	$+\sqrt{5}/8\sqrt{7}$
1.1	1.21	2.1^2	0	−	$-\sqrt{5}/8\sqrt{7}$
1.1	1.3^2	0.2	0	+	$-1/4\sqrt{2.7}$
1.1	3.21	2.1^2	0	+	$-1/2\sqrt{5.7}$
1.2^2	1.0	0.2	0	+	$+1/4\sqrt{5}$
1.2^2	1.21	0.2	0	+	$-1/4\sqrt{2}$
1.2^2	1.21	2.1^2	0	+	$+3/4\sqrt{2.5}$
1.2^2	3.21	2.1^2	0	−	0
1.31	1.21	0.2	0	−	$-\sqrt{3}/8$
1.31	1.21	2.1^2	0	−	$+1/8\sqrt{3}$
1.31	1.3^2	0.2	0	+	$-\sqrt{3}/4\sqrt{2}$
1.31	1.3^2	2.1^2	0	+	$-1/4\sqrt{3}$
1.31	3.21	2.1^2	0	+	$-1/2\sqrt{3}$
3.1	1.0	2.1^2	0	+	0
3.1	1.21	2.1^2	0	+	$-1/2\sqrt{2.5}$
3.1	3.21	0.2	0	−	$+1/4\sqrt{5}$
3.1	3.21	2.1^2	0	−	$+1/4$
3.31	1.21	2.1^2	0	+	$-1/2\sqrt{2.3}$
3.31	1.3^2	2.1^2	0	−	$+1/\sqrt{2.3}$
3.31	3.21	0.2	0	−	$+\sqrt{3}/4$
3.31	3.21	2.1^2	0	−	$+\sqrt{5}/4\sqrt{3}$
31^4	**2^41**	**2**	**0**		
1.1	1.0	0.1^2	0	−	$+1/2\sqrt{2.5.7}$
1.1	1.21	0.1^2	0	−	$+3/4\sqrt{2.5.7}$
1.1	1.21	2.2	0	−	$-19/4.3\sqrt{2.5.7}$
1.1	1.3^2	2.2	0	+	$-1/4.3\sqrt{7}$
1.1	3.21	2.2	0	+	$+1/3\sqrt{2.5.7}$
1.2^2	1.0	2.2	0	+	$-1/2\sqrt{2.5}$
1.2^2	1.21	0.1^2	0	+	$-1/4\sqrt{5}$
1.2^2	1.21	2.2	0	+	$+1/4$
1.2^2	3.21	2.2	0	−	0
1.31	1.21	0.1^2	0	−	$+1/4\sqrt{2.3}$
1.31	1.21	2.2	0	−	$-1/4\sqrt{2.3}$
1.31	1.3^2	0.1^2	0	+	$-1/2\sqrt{2.3}$

Table 12—*continued*

31^4	**2^41**	**2**	**0**		
1.31	1.3^2	2.2	0	+	$-1/4\sqrt{3}$
1.31	3.21	2.2	0	+	$+1/\sqrt{2.3}$
3.1	1.21	2.2	0	+	$+1/2.3\sqrt{5}$
3.1	1.3^2	2.2	0	−	$+1/3\sqrt{2}$
3.1	3.21	0.1^2	0	−	$-1/2\sqrt{2.5}$
3.1	3.21	2.2	0	−	$-1/2.3\sqrt{2}$
3.31	1.21	2.2	0	+	$+1/2\sqrt{3}$
3.31	1.3^2	2.2	0	−	$+1/\sqrt{2.3}$
3.31	3.21	0.1^2	0	−	$-1/2\sqrt{2.3}$
3.31	3.21	2.2	0	−	$-\sqrt{5}/2\sqrt{2.3}$
31^4	**21^3**	**21^4**	**0**		
1.1	1.1^2	0.21	0	−	$-i/2\sqrt{5.7}$
1.1	1.1^2	2.0	0	+	$+\sqrt{3}/4\sqrt{2.5.7}$
1.1	1.1^2	2.21	0	+	$+\sqrt{5}/8.2\sqrt{3.7}$
1.1	1.2	0.21	0	+	$+i/8\sqrt{2.3.7}$
1.1	1.2	2.21	0	−	$+1/3\sqrt{2.7}$
1.1	1.32	0.21	0	−	$+i\sqrt{5}/8\sqrt{7}$
1.1	1.32	2.21	0	+	$+\sqrt{5}/8.2\sqrt{3.7}$
1.1	3.1^2	2.0	0	−	$-1/2\sqrt{2.3.7}$
1.1	3.1^2	2.21	0	−	$-\sqrt{3}/8\sqrt{7}$
1.1	3.2	2.21	0	+	$-5/8.3\sqrt{7}$
1.2^2	1.1^2	0.21	0	+	$+i/8\sqrt{2.5}$
1.2^2	1.1^2	2.21	0	−	$+1/4\sqrt{2.3.5}$
1.2^2	1.2	0.21	0	−	0
1.2^2	1.2	2.0	0	+	$-1/4.3$
1.2^2	1.2	2.21	0	+	$+\sqrt{5}/8.3$
1.2^2	1.32	0.21	0	+	$-i\sqrt{5}/8\sqrt{2}$
1.2^2	1.32	2.21	0	−	0
1.2^2	3.1^2	2.21	0	+	$-1/4\sqrt{2.3}$
1.2^2	3.2	2.0	0	−	$+1/2.3\sqrt{2}$
1.2^2	3.2	2.21	0	−	$-\sqrt{5}/4.3\sqrt{2}$
1.31	1.1^2	0.21	0	−	$-i/8\sqrt{3}$
1.31	1.1^2	2.21	0	+	$-7/8.2.3$
1.31	1.2	0.21	0	+	$+i\sqrt{5}/8\sqrt{2}$
1.31	1.2	2.21	0	−	$+\sqrt{5}/4\sqrt{2.3}$
1.31	1.32	0.21	0	−	0
1.31	1.32	0.21	1	+	0
1.31	1.32	2.0	0	+	$-\sqrt{5}/4.3\sqrt{2}$
1.31	1.32	2.21	0	+	$-5\sqrt{5}/8.2.3$
1.31	1.32	2.21	1	−	$-i\sqrt{5}/8\sqrt{3}$
1.31	3.1^2	2.21	0	−	$-\sqrt{5}/8.3$
1.31	3.2	2.21	0	+	$+\sqrt{5}/8\sqrt{3}$
3.1	1.1^2	2.0	0	−	$+1/4\sqrt{3.5}$
3.1	1.1^2	2.21	0	−	$+7/8\sqrt{2.3.5}$
3.1	1.2	2.21	0	+	$-1/8.3$
3.1	1.32	2.21	0	−	$+\sqrt{5}/8\sqrt{2.3}$
3.1	3.1^2	0.21	0	−	$-i/8$
3.1	3.1^2	2.0	0	+	0
3.1	3.1^2	2.21	0	+	$-\sqrt{5}/8\sqrt{3}$
3.1	3.2	0.21	0	+	$+i/8\sqrt{3}$
3.1	3.2	2.21	0	−	$+\sqrt{5}/8.3$
3.31	1.1^2	2.21	0	−	$-1/8.3\sqrt{2}$
3.31	1.2	2.21	0	+	$-\sqrt{5}/8\sqrt{3}$
3.31	1.32	2.0	0	−	$-\sqrt{5}/4.3$
3.31	1.32	2.21	0	−	$-5\sqrt{5}/8.3\sqrt{2}$
31^4	**21^3**	**21^4**	**0**		
3.31	1.32	2.21	1	+	$-i\sqrt{5}/4\sqrt{2.3}$
3.31	3.1^2	0.21	0	−	$-i\sqrt{5}/8\sqrt{3}$
3.31	3.1^2	2.21	0	+	$-5/8.3$
3.31	3.2	0.21	0	+	$+i\sqrt{5}/8$
3.31	3.2	2.21	0	−	$+5/8\sqrt{3}$
31^4	**2^21^3**	**1^4**	**0**		
1.1	1.1	0.2^2	0	+	$-5/8.2\sqrt{7}$
1.1	1.1	2.1	0	+	$+5/8\sqrt{2.3.7}$
1.1	1.2^2	2.1	0	−	$-11/8\sqrt{2.3.5.7}$
1.1	1.31	0.2^2	0	+	$-\sqrt{3}/8.2\sqrt{7}$
1.1	3.1	2.1	0	−	$-1/2\sqrt{2.3.5.7}$
1.1	3.2^2	2.1	0	+	$-\sqrt{5}/4\sqrt{3.7}$
1.2^2	1.1	2.1	0	−	$-1/8\sqrt{2}$
1.2^2	1.2^2	0.2^2	0	+	$-3/8\sqrt{5}$
1.2^2	1.31	0.2^2	0	−	$-1/8$
1.2^2	1.31	2.1	0	−	$-\sqrt{3}/8\sqrt{2}$
1.2^2	3.1	2.1	0	+	$-1/2\sqrt{2.5}$
1.31	1.1	0.2^2	0	+	$-\sqrt{5}/8.2$
1.31	1.2^2	0.2^2	0	−	$-1/8$
1.31	1.2^2	2.1	0	−	$+5/8\sqrt{2.3}$
1.31	1.31	0.2^2	0	+	$+3\sqrt{3}/8.2$
1.31	1.31	2.1	0	+	$-\sqrt{3}/8\sqrt{2}$
1.31	3.2^2	2.1	0	+	$-1/4\sqrt{3}$
3.1	1.1	2.1	0	−	$+1/4\sqrt{3}$
3.1	1.2^2	2.1	0	+	$-1/4\sqrt{3.5}$
3.1	3.1	0.2^2	0	+	$-1/4\sqrt{5}$
3.1	3.1	2.1	0	+	$-1/2\sqrt{2.3}$
3.1	3.2^2	2.1	0	−	$+1/4\sqrt{3}$
3.31	1.2^2	2.1	0	+	$+1/4\sqrt{3}$
3.31	1.31	2.1	0	−	$+\sqrt{3}/4$
3.31	3.1	0.2^2	0	+	$+1/4$
3.31	3.2^2	0.2^2	0	−	$+1/2\sqrt{2}$
3.31	3.2^2	2.1	0	−	$-\sqrt{5}/4\sqrt{3}$
31^4	**2^21^3**	**2^5**	**0**		
1.1	1.1	0.1	0	+	$-1/8\sqrt{2.5}$
1.1	1.1	2.2^2	0	+	$+1/8.2\sqrt{3.5}$
1.1	1.2^2	0.1	0	−	$-1/8\sqrt{2}$
1.1	1.31	2.2^2	0	+	$+\sqrt{5}/8.2$
1.1	3.1	2.2^2	0	−	$-1/4\sqrt{3}$
1.2^2	1.1	0.1	0	−	$-\sqrt{3}/8\sqrt{2.5.7}$
1.2^2	1.2^2	2.2^2	0	+	$-\sqrt{3}/8\sqrt{7}$
1.2^2	1.31	0.1	0	−	$+\sqrt{5}/8\sqrt{2.7}$
1.2^2	1.31	2.2^2	0	−	$+\sqrt{3.5}/8\sqrt{7}$
1.2^2	3.2^2	2.2^2	0	−	$-\sqrt{3}/2\sqrt{2.7}$
1.31	1.1	2.2^2	0	+	$-13/8.2\sqrt{3.7}$
1.31	1.2^2	0.1	0	−	$+\sqrt{5}/8\sqrt{2.7}$
1.31	1.2^2	2.2^2	0	−	$-5\sqrt{5}/8\sqrt{3.7}$
1.31	1.31	0.1	0	+	$-3\sqrt{5}/8\sqrt{2.7}$
1.31	1.31	2.2^2	0	+	$+3\sqrt{5}/8.2\sqrt{7}$
1.31	3.1	2.2^2	0	−	$+\sqrt{5}/4\sqrt{3.7}$
1.31	3.2^2	2.2^2	0	+	$+\sqrt{5}/2\sqrt{2.3.7}$
3.1	1.1	2.2^2	0	−	$+11/4\sqrt{2.3.5.7}$
3.1	1.31	2.2^2	0	−	$-\sqrt{5}/4\sqrt{2.7}$
3.1	3.1	0.1	0	+	$-1/2\sqrt{2.7}$
3.1	3.1	2.2^2	0	+	$-\sqrt{5}/4\sqrt{3.7}$

Table 12—*continued*

31^4	**2^21^3**	**2^5**	**0**		
3.1	3.2^2	0.1	0	−	$+1/4\sqrt{7}$
3.31	1.1	2.2^2	0	−	$+1/4\sqrt{2.3.7}$
3.31	1.2^2	2.2^2	0	+	$-\sqrt{5}/2\sqrt{2.3.7}$
3.31	1.31	2.2^2	0	−	$-3\sqrt{5}/4\sqrt{2.7}$
3.31	3.1	2.2^2	0	+	$+5/4\sqrt{3.7}$
3.31	3.2^2	0.1	0	−	$-\sqrt{5}/4\sqrt{7}$
3.31	3.2^2	2.2^2	0	−	$+5/2\sqrt{2.3.7}$
31^4	**3^5**	**1^2**	**0**		
1.1	1.21	0.2	0	−	$+1/2\sqrt{2.5}$
1.1	1.21	2.1^2	0	−	$-1/2\sqrt{2.5}$
1.2^2	1.21	0.2	0	+	$-1/2\sqrt{7}$
1.2^2	1.21	2.1^2	0	+	$+3/2\sqrt{5.7}$
1.31	1.21	0.2	0	−	$+\sqrt{3}/2\sqrt{2.7}$
1.31	1.21	2.1^2	0	−	$-1/2\sqrt{2.3.7}$
1.31	3.3^2	2.1^2	0	−	$-2/\sqrt{3.7}$
3.1	1.21	2.1^2	0	+	$+1/\sqrt{5.7}$
3.1	3.3^2	0.2	0	+	$-1/\sqrt{2.7}$
3.31	1.21	2.1^2	0	+	$+1/\sqrt{3.7}$
3.31	3.3^2	0.2	0	+	$-\sqrt{3}/\sqrt{2.7}$
3.31	3.3^2	2.1^2	0	+	$+\sqrt{5}/\sqrt{3.7}$
31^4	**3^5**	**2**	**0**		
1.1	1.21	0.1^2	0	−	$+1/2\sqrt{2.3.5.7}$
1.1	1.21	2.2	0	−	$-17/2.3\sqrt{2.3.5.7}$
1.1	3.3^2	2.2	0	−	$-\sqrt{2}/3\sqrt{3.7}$
1.2^2	1.21	0.1^2	0	+	$-1/2\sqrt{3.5}$
1.2^2	1.21	2.2	0	+	$+1/2\sqrt{3}$
1.31	1.21	0.1^2	0	−	$-1/2.3\sqrt{2}$
1.31	1.21	2.2	0	−	$+1/2.3\sqrt{2}$
1.31	3.3^2	2.2	0	−	$-\sqrt{2}/3$
3.1	1.21	2.2	0	+	$-1/3\sqrt{3.5}$
3.1	3.3^2	2.2	0	+	$+\sqrt{5}/3\sqrt{2.3}$
3.31	1.21	2.2	0	+	$-1/3$
3.31	3.3^2	0.1^2	0	+	$-1/3$
3.31	3.3^2	2.2	0	+	$+\sqrt{5}/3\sqrt{2}$
31^4	**31^4**	**1^4**	**0**		
1.1	1.1	0.2^2	0	+	$+\sqrt{3}/8\sqrt{2.5}$
1.1	1.1	2.1	0	+	$+1/8.3\sqrt{5}$
1.2^2	1.1	2.1	0	−	$-\sqrt{7}/8\sqrt{3.5}$
1.2^2	1.2^2	0.2^2	0	+	$-1/4\sqrt{2.5}$
1.31	1.1	0.2^2	0	+	$-\sqrt{7}/8\sqrt{2.3}$
1.31	1.2^2	0.2^2	0	−	$+1/4\sqrt{2}$
1.31	1.2^2	2.1	0	−	$-1/8\sqrt{3}$
1.31	1.31	0.2^2	0	+	$-\sqrt{3}/8\sqrt{2}$
1.31	1.31	2.1	0	+	$+5/8\sqrt{3}$
3.1	1.1	2.1	0	−	$-\sqrt{7}/2.3\sqrt{2.5}$
3.1	1.2^2	2.1	0	+	$-1/2\sqrt{2.3.5}$
3.1	3.1	0.2^2	0	+	$+\sqrt{3}/4\sqrt{5}$
3.1	3.1	2.1	0	+	$+1/2.3\sqrt{2}$
3.31	1.2^2	2.1	0	+	$+1/2\sqrt{2.3}$
3.31	1.31	2.1	0	−	$+1/2\sqrt{2.3}$
3.31	3.1	0.2^2	0	+	$+1/4\sqrt{3}$
3.31	3.31	0.2^2	0	+	$+\sqrt{3}/4$
3.31	3.31	2.1	0	+	$-\sqrt{5}/2\sqrt{2.3}$
31^4	**31^4**	**2^5**	**0**		
1.1	1.1	0.1	0	+	$+11/8.3\sqrt{3.5.7}$
31^4	**31^4**	**2^5**	**0**		
1.1	1.1	2.2^2	0	+	$-13/8.3\sqrt{2.5.7}$
1.2^2	1.1	0.1	0	−	$+1/8.3\sqrt{5}$
1.2^2	1.2^2	2.2^2	0	+	$+\sqrt{7}/4\sqrt{2.3.5}$
1.31	1.1	2.2^2	0	+	$-11/8.9\sqrt{2}$
1.31	1.2^2	0.1	0	−	$-\sqrt{7}/8.3$
1.31	1.2^2	2.2^2	0	−	$+\sqrt{7}/4.3\sqrt{2.3}$
1.31	1.31	0.1	0	+	$+\sqrt{7}/8.3$
1.31	1.31	2.2^2	0	+	$-5\sqrt{7}/8.3\sqrt{2}$
3.1	1.1	2.2^2	0	−	$-1/4\sqrt{5}$
3.1	1.31	2.2^2	0	−	$-\sqrt{7}/4.9$
3.1	3.1	0.1	0	+	$+\sqrt{7}/2.3\sqrt{2.3.5}$
3.1	3.1	2.2^2	0	+	$+\sqrt{7}/4.3$
3.31	1.1	2.2^2	0	−	$+5/4.9$
3.31	1.2^2	2.2^2	0	+	$-\sqrt{7}/2.3\sqrt{3}$
3.31	1.31	2.2^2	0	−	$-\sqrt{7}/4.3$
3.31	3.1	2.2^2	0	+	$+\sqrt{5.7}/4.9$
3.31	3.31	0.1	0	+	$-\sqrt{7}/2.3\sqrt{2}$
3.31	3.31	2.2^2	0	+	$+\sqrt{5.7}/4.3$
32^4	**31^4**	**0**	**0**		
1.1^2	1.1	0.0	0	+	$+1/2\sqrt{5}$
1.2	1.2^2	0.0	0	+	$+1/\sqrt{2.5}$
1.32	1.31	0.0	0	+	$+1/2$
3.1^2	3.1	0.0	0	+	$+1/\sqrt{2.5}$
3.32	3.31	0.0	0	+	$+1/\sqrt{2}$
32^4	**31^4**	**21^4**	**0**		
1.1^2	1.1	0.21	0	−	$+47i/8.2.3\sqrt{5.7.11}$
1.1^2	1.1	2.0	0	+	$-31/4.9\sqrt{2.3.5.7.11}$
1.1^2	1.1	2.21	0	+	$+8.2/9\sqrt{3.5.7.11}$
1.1^2	1.2^2	0.21	0	+	$-i\sqrt{5}/4.3\sqrt{2.11}$
1.1^2	1.2^2	2.21	0	−	$+19/8.3\sqrt{2.3.5.11}$
1.1^2	1.31	0.21	0	−	$-5i/8.2.3\sqrt{3.11}$
1.1^2	1.31	2.21	0	+	$+3/8\sqrt{11}$
1.1^2	3.1	2.0	0	−	$-\sqrt{5}/4.9\sqrt{3.11}$
1.1^2	3.1	2.21	0	−	$-157/8.9\sqrt{2.3.5.11}$
1.1^2	3.31	2.21	0	−	$-13/8.3\sqrt{2.11}$
1.2	1.2^2	0.21	0	−	$-i\sqrt{7}/8\sqrt{11}$
1.2	1.2^2	2.0	0	+	$-\sqrt{7}/4.3\sqrt{3.5.11}$
1.2	1.2^2	2.21	0	+	$-\sqrt{7}/4.3\sqrt{3.11}$
1.2	1.31	0.21	0	+	0
1.2	1.31	2.21	0	−	$-7\sqrt{7}/8.3\sqrt{2.11}$
1.2	3.1	2.21	0	+	$+7\sqrt{7}/8.3\sqrt{3.5.11}$
1.2	3.31	2.21	0	+	$+7\sqrt{7}/8.3\sqrt{11}$
1.32	1.31	0.21	0	−	$+i\sqrt{7.11}/8.2.3$
1.32	1.31	0.21	1	+	$+\sqrt{3.7}/8\sqrt{11}$
1.32	1.31	2.0	0	+	$-13\sqrt{7}/4.9\sqrt{2.3.11}$
1.32	1.31	2.21	0	+	$-17\sqrt{7}/4.9\sqrt{3.11}$
1.32	1.31	2.21	1	−	$+2i\sqrt{7}/9\sqrt{11}$
1.32	3.1	2.21	0	−	$+\sqrt{7}/8.3\sqrt{2.11}$
1.32	3.31	2.0	0	−	$-5\sqrt{7}/4.9\sqrt{3.11}$
1.32	3.31	2.21	0	−	$-31\sqrt{7}/8.9\sqrt{2.3.11}$
1.32	3.31	2.21	1	+	$+i\sqrt{7.11}/4.9\sqrt{2}$
3.1^2	3.1	0.21	0	−	$+i\sqrt{7.11}/8.3\sqrt{2.5}$
3.1^2	3.1	2.0	0	+	$-2\sqrt{7}/9\sqrt{3.11}$
3.1^2	3.1	2.21	0	+	$-37\sqrt{7}/8.9\sqrt{2.3.11}$
3.1^2	3.31	0.21	0	−	$-5i\sqrt{7}/8.3\sqrt{2.3.11}$

Table 12—*continued*

32^4	**31^4**	**21^4**	**0**		
3.1^2	3.31	2.21	0	+	$+\sqrt{5.7}/8\sqrt{2.11}$
3.32	3.31	0.21	0	−	$+i\sqrt{7.11}/8.3\sqrt{2}$
3.32	3.31	0.21	1	+	$+\sqrt{3.7}/4\sqrt{2.11}$
3.32	3.31	2.0	0	+	$-2\sqrt{5.7}/9\sqrt{3.11}$
3.32	3.31	2.21	0	+	$-37\sqrt{5.7}/8.9\sqrt{2.3.11}$
3.32	3.31	2.21	1	−	$+5i\sqrt{5.7}/4.9\sqrt{2.11}$
32^4	**31^4**	**21^4**	**1**		
1.1^2	1.1	0.21	0	+	$-\sqrt{7}/8\sqrt{2.5.11}$
1.1^2	1.1	2.0	0	−	$-i\sqrt{7}/4\sqrt{3.5.11}$
1.1^2	1.1	2.21	0	−	$+i\sqrt{7}/2\sqrt{2.3.5.11}$
1.1^2	1.2^2	0.21	0	−	$-3/4\sqrt{5.11}$
1.1^2	1.2^2	2.21	0	+	$+i\sqrt{3.5}/8\sqrt{11}$
1.1^2	1.31	0.21	0	+	$-\sqrt{3}/8\sqrt{2.11}$
1.1^2	1.31	2.21	0	−	$-i/4\sqrt{2.11}$
1.1^2	3.1	2.0	0	+	$+i/2\sqrt{2.3.5.11}$
1.1^2	3.1	2.21	0	+	$+i\sqrt{5}/8\sqrt{3.11}$
1.1^2	3.31	2.21	0	+	$-i/8\sqrt{11}$
1.2	1.2^2	0.21	0	+	$+5/4\sqrt{2.7.11}$
1.2	1.2^2	2.0	0	−	$-3i\sqrt{3}/2\sqrt{2.5.7.11}$
1.2	1.2^2	2.21	0	−	$-3i\sqrt{3}/2\sqrt{2.7.11}$
1.2	1.31	0.21	0	−	0
1.2	1.31	2.21	0	+	$+3i/8\sqrt{7.11}$
32^4	**31^4**	**21^4**	**1**		
1.2	3.1	2.21	0	−	$-i\sqrt{3}/4\sqrt{2.5.7.11}$
1.2	3.31	2.21	0	−	$-3i/4\sqrt{2.7.11}$
1.32	1.31	0.21	0	+	$+\sqrt{11}/8\sqrt{2.7}$
1.32	1.31	0.21	1	−	$+5i\sqrt{3}/4\sqrt{2.7.11}$
1.32	1.31	2.0	0	−	$+5i/4\sqrt{3.7.11}$
1.32	1.31	2.21	0	−	$+i\sqrt{2}/\sqrt{3.7.11}$
1.32	1.31	2.21	1	+	$-1/\sqrt{2.7.11}$
1.32	3.1	2.21	0	+	$-13i/8\sqrt{7.11}$
1.32	3.31	2.0	0	+	$+i\sqrt{7}/2\sqrt{2.3.11}$
1.32	3.31	2.21	0	+	$+17i/8\sqrt{3.7.11}$
1.32	3.31	2.21	1	−	$-\sqrt{11}/4\sqrt{7}$
3.1^2	3.1	0.21	0	+	$+\sqrt{11}/8\sqrt{5.7}$
3.1^2	3.1	2.0	0	−	$-i/\sqrt{2.3.7.11}$
3.1^2	3.1	2.21	0	−	$-i/8\sqrt{3.7.11}$
3.1^2	3.31	0.21	0	+	$-\sqrt{3.7}/8\sqrt{11}$
3.1^2	3.31	2.21	0	−	$+5i\sqrt{5}/8\sqrt{7.11}$
3.32	3.31	0.21	0	+	$+\sqrt{11}/8\sqrt{7}$
3.32	3.31	0.21	1	−	$+5i\sqrt{3}/4\sqrt{7.11}$
3.32	3.31	2.0	0	−	$-i\sqrt{5}/\sqrt{2.3.7.11}$
3.32	3.31	2.21	0	−	$-i\sqrt{5}/8\sqrt{3.7.11}$
3.32	3.31	2.21	1	+	$+\sqrt{5.7}/4\sqrt{11}$

orthogonal nature of some SU_3 irreps and the symplectic nature of some SU_2 irreps means that many $2jm$'s are -1. From table 9, line 6 we have

$$\begin{pmatrix}21\\0.0\end{pmatrix}=\begin{pmatrix}21\\3.1\end{pmatrix}=\begin{pmatrix}21\\0.2\end{pmatrix}=1 \qquad \text{and}\begin{pmatrix}21\\-3.1\end{pmatrix}=-1. \tag{3.4}$$

$3jm$ factors are zero unless the top and bottom rows are triads of the group and subgroups respectively, and unless the columns obey the branching rules. The symmetries of the $3jm$ are: invariance under cyclic permutations of the columns; a possible sign change for a column interchange, the (12) interchange being

$$\begin{pmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ \rho_1 & \rho_2 & \rho_3\end{pmatrix}^{r}_{s}=\pm\begin{pmatrix}\lambda_2 & \lambda_1 & \lambda_3\\ \rho_2 & \rho_1 & \rho_3\end{pmatrix}^{r}_{s}, \tag{3.5}$$

and a possible sign change under the complex conjugation symmetry

$$=\pm\begin{pmatrix}\lambda_1^* & \lambda_2^* & \lambda_3^*\\ \rho_1^* & \rho_2^* & \rho_3^*\end{pmatrix}^{r^*}_{s}. \tag{3.6}$$

For $SU_3 \supset U_1 \times SU_2$ and $SU_6 \supset SU_2 \times SU_3$ the irrep labels for the direct product subgroup are a product of the labels for each group, and for the second case, the subgroup coupling multiplicity label refers to the multiplicity in SU_3.

The $3jm$ tables (tables 8, 10 and 12) use the top row of a $3jm$ as a bold typeface header, each subsequent entry giving the subgroup irrep label pairs, the label s (for table 12 only), the column interchange sign (for equation (3.5)), a star if complex conjugation introduces a negative sign (for equation (3.6)), and then the value. The symmetries are used to reduce the size of the tables; the irreps appear in the order of tables 1 and 2.

4. The symmetric group–unitary group duality

The relationship between the structures of the symmetric and unitary groups was recognised by Frobenius and Schur. Weyl (1946) makes much use of this duality, showing that irreps of the unitary groups can be obtained using Young symmetrisers. Both the symmetric and unitary group characters are specified by Schur functions (Littlewood 1950), which were studied by Jacobi, Trudi, Kostka and others long before Schur (1901) showed the connection with the characters of these groups. The use of the purely functional combinatoric properties of Schur functions has recently proved fruitful in obtaining new identities, and thus new computational techniques for character theory (see for example Butler and King (1973a, b), King (1970, 1975), Wybourne (1970)). The algebra of Schur functions makes the dual structures of the characters of S_l and U_n apparent.

The duality goes further than character theory, and one can establish many identities between the Racah–Wigner algebra of S_l and that of U_n. Nuclear shell model theorists used this duality to compute jm and j symbols of U_n (Jahn 1954, Elliott *et al* 1953, Kaplan 1962a, b, Horie 1964, Vanagas 1971). Kramer (1967, 1968) obtained the equality of 'f-symbols' for symmetric group chains with j symbols of all unitary groups and jm factors in $U_{p+q} \supset U_p \times U_q$ bases. These results yield the Regge symmetries for the $3jm$ symbols of $SU_2 \supset U_1$ (Kramer and Seligman 1969a).

Another approach to duality applies the concept of double coset (DC) generators (Kramer and Seligman 1969b) to relate the matrix elements of double coset generators (DCME) to $9f$ symbols of a certain symmetric group chains and hence to appropriate $9j$ symbols of any unitary group. Sullivan (1976, 1980, and references therein) has formulated the general problem of DC decompositions and arrived at more duality results.

None of these calculations is complete because phase and multiplicity relationships between different unitary and symmetric groups have been left unspecified. In addition, simplifications can be obtained by using the isomorphism between U_n and $U_1 \times SU_n$. The coefficients of U_n calculated by Baird and Biedenharn (1964) and So and Strottman (1979) are not factorisable into $U_1 \times SU_n$ coefficients. The application of the tilde symmetry for S_l (see Butler and Ford 1979) provides further symmetries for U_n coefficients, in analogy to the SU_2 Regge symmetries. A full discussion of these symmetries must include duality phases.

Although the phase questions remain unsolved, our tables provide illustrations of the dualities. The SU_6 $6j$ $\begin{Bmatrix} 21 & 2^5 & 1^5 \\ 1 & 2 & 1^5 \end{Bmatrix}$ may be found in table 6, as 1/2.3.7. Using the $6j$ symmetries, in particular (13) column interchange and complex conjugation, this becomes

$$\left|\begin{Bmatrix} 21 & 2^5 & 1^5 \\ 1 & 2 & 1^5 \end{Bmatrix}_{SU_6}\right| = \left|\begin{Bmatrix} 1 & 2 & 21^* \\ 1^* & 2 & 1 \end{Bmatrix}_{SU_6}\right|.$$

Writing this as a recoupling coefficient (Butler and Wybourne 1976, equation (13)–note that there is an error in the ordering of subscripts in the $6j$ symbol on the right-hand side of this equation) and using Kramer (1967, equation (6.14)) allows us to change unitary groups. From the SU_3 $6j$ table we have

$$= \frac{|2|_{SU_3}}{|2|_{SU_6}} \left|\begin{Bmatrix} 1 & 2 & 21^* \\ 1^* & 2 & 1 \end{Bmatrix}_{SU_3}\right| = \frac{6}{21} \left|\begin{Bmatrix} 21 & 2 & 1 \\ 1^2 & 2^* & 1 \end{Bmatrix}_{SU_3}\right| = \frac{6}{21} \cdot \frac{1}{2.6}$$

while from the SU_2 table (Rotenberg *et al* 1959) this is also

$$=\frac{|2|_{SU_2}}{|2|_{SU_6}}\left|\begin{Bmatrix} 1 & 2 & 21^* \\ 1^* & 2 & 1 \end{Bmatrix}_{SU_2}\right| = \frac{3}{21}\left|\begin{Bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \end{Bmatrix}_{SU_2}\right| = \frac{3}{21}\cdot\frac{1}{2.3}.$$

In a similar fashion the coupling (or isoscalar) factors of SU_{pq} in the $SU_p \times SU_q$ basis, are p and q independent, (see for example Sullivan (1980), Chen (1981)). The $3jm$

$$\begin{pmatrix} 1^3 & 1^2 & 1 \\ 1.21 & 0.2 & 1.1 \end{pmatrix}^{SU_6}_{SU_2\times SU_3} = \sqrt{\frac{2}{5}}$$

as found in table 12, may be successively transformed into a trivial $3jm$ of $SU_4 \supset SU_2 \times SU_2$:

$$\left|\begin{pmatrix} 1^3 & 1^2 & 1 \\ 1.21 & 0.2 & 1.1 \end{pmatrix}^{SU_6}_{SU_2\times SU_3}\right|$$

$$= \left|\begin{pmatrix} 1 & 1^3 & 1^2 \\ 1.1 & 1.21 & 0.2 \end{pmatrix}^{SU_6}_{SU_2\times SU_3}\right|$$

$$= \left(\frac{|0|_{SU_2}|2|_{SU_3}}{|1^2|_{SU_6}}\right)^{1/2}\left|\begin{pmatrix} 1 & 1^3 & 1^4 \\ 1.1 & 21.21 & 2^2.2^2 \end{pmatrix}^{U_{pq}}_{U_p\times U_q}\right|$$

$$= \left(\frac{|0|_{SU_2}|2|_{SU_3}}{|1^2|_{SU_6}}\cdot\frac{|0|_{SU_4}}{|0|_{SU_2}|0|_{SU_2}}\right)^{1/2}\left|\begin{pmatrix} 1 & 1^3 & 0 \\ 1.1 & 1.1 & 0.0 \end{pmatrix}^{SU_4}_{SU_2\times SU_2}\right|.$$

Trivial $3jm$'s are just a ratio of dimensions, in this case unity.

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